BMS INSTITUTE OF TECHNOLOGY AND MANAGEMENT

YELAHANKA – BANGALORE - 64

DEPARTMENT OF ELECTRONICS & TELECOMMUNICATION ENGINEERING

Course Name:	**Signal Processing**
Course Code:	**18EC55**
Semester :	**3rd semester**
Prepared by :	**Prof. Thejaswini S**

CONTENTS

Books referred:

1. Signals & System by Simon Haykins
2. Signals & System by Uday kumar S
3. Signals & System by Ganesh Rao

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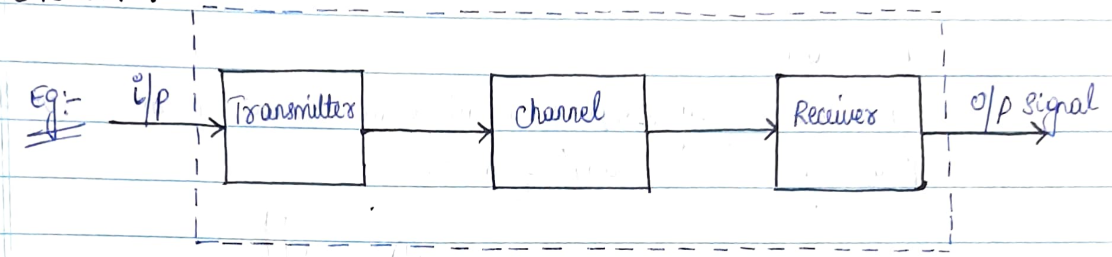
MODULE-1

INTRODUCTION:

*** SIGNALS:-

A Signal is formally defined as a function of one or more variables which conveys information on the nature of physical phenomenon.

SYSTEM :-



COMMUNICATION SYSTEM

*** A System is formally defined as an entity that manipulates one or more signals to accomplish a function there by yielding a new signal.

*** Examples of control Systems.

- Open loop control system – washing machine, micro-wave.
- closed loop control system – Iron-Box.

2/02/18 *CLASSIFICATION OF SIGNALS:-

(i) CONTINUOUS / DISCRETE TIME SIGNALS
~~(ii)~~ ANALOG / DIGITAL SIGNALS.
(iii) EVEN / ODD SIGNALS
(iv) DETERMINISTIC & RANDOM SIGNALS
(v) PERIODIC & APERIODIC / NON-PERIODIC SIGNALS
(vi) ENERGY & POWER SIGNALS

i) CONTINUOUS & DISCRETE TIME SIGNALS (CT & DT)

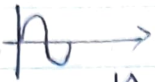
CT – The Signal is defined for all values of 't'
– duration $(-\infty, \infty)$.

If CT are defined b/w some interval (a,b) — Analog signal.

REPRESENTATION OF CT SIGNALS:-

(i) $x(t) = t^2$ or $x(t) = \sin t$ or $x(t) = e^{-t}$ — FORMULA

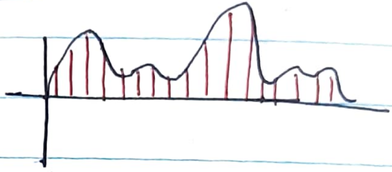
(ii) GRAPHICAL

- Sine 
- Triangular 
- Square 

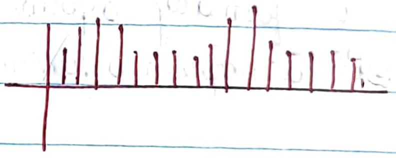
DT Signals

- Are obtained by sampling CT at equal intervals of time with Sampling frequency f_s & Sampling time T.

Eg: Take a CT signal



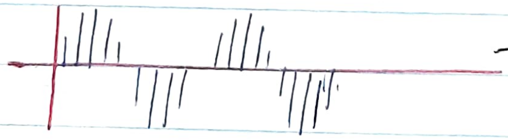
DT Signal



Representation of DT:-

① $x(n) = n^2$
$x(n) = e^{-n^2}$ } FORMULA

② Graphical.



— Sinusoidal DT.

CT — $x(t)$
DT — $x(n)$

RELATION B/W CT & DT:-

$$x(t) = x(nT) = x(t + nT)$$

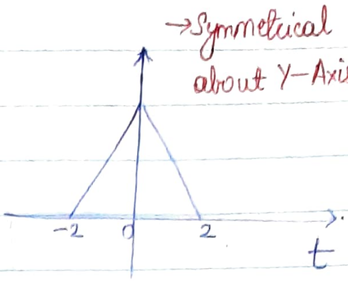
(3) EVEN & ODD SIGNALS :

EVEN SIGNAL

CONTINUOUS TIME SIGNALS :-

A $x(t)$ is said to be even signal iff

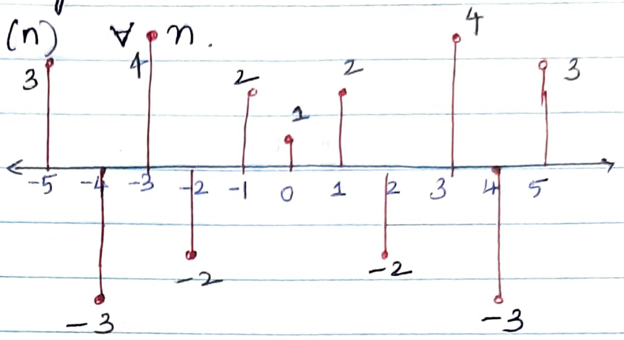
$$x(-t) = x(t) \quad \forall\ t$$



DISCRETE TIME SIGNALS :-

A $x(n)$ is said to be even signal iff

$$x(-n) = x(n) \quad \forall\ n.$$



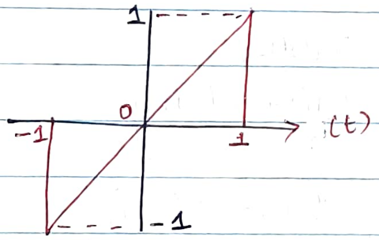
***

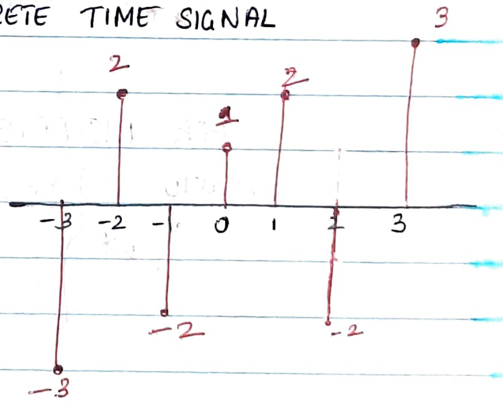
EVEN SIGNALS ARE SYMMETRIC ABOUT Y-AXIS.

ODD SIGNAL:

$x(-t) = -x(t) \quad \forall\ t$ — CONTINUOUS TIME SIGNAL

$x(-n) = -x(n) \quad \forall\ n$ — DISCRETE TIME SIGNAL





① Develop even/odd decompositions of a general signal $x(t)$ be applying the definitions.

A ① $x(t)$ — CTS.

$$\boxed{x(t) = x_e(t) + x_o(t)} \quad \text{— ①}$$

where $x_e(t)$ = even part of $x(t)$.

$x_o(t)$ = odd part of $x(t)$.

replace t by -t in equation ①

$x(-t) = x_e(-t) + x_o(-t)$ — ②

COND^n $\{ x_e(-t) = x_e(t)$ & $x_o(-t) = -x_o(t)$

$\boxed{x(-t) = x_e(t) - x_o(t)}$ — ③

Using : Add ① & ③

$x(t) + x(-t) = x_e(t) + \cancel{x_o(t)} + x_e(t) - \cancel{x_o(t)}$

$x(t) + x(-t) = 2x_e(t)$

*** $\boxed{x_e(t) = \frac{1}{2}\{x(t) + x(-t)\}}$ — ④

Subtracting ① & ③

$x(t) - x(-t) = \cancel{x_e(t)} + x_o(t) - \cancel{x_e(t)} + x_o(t)$

$x(t) - x(-t) = 2x_o(t)$

*** $\boxed{x_o(t) = \frac{1}{2}\{x(t) - x(-t)\}}$ — ⑤

FOR DISCRETE TIME SIGNALS :-

Similarly

$$\left\{\begin{array}{l} x_e(n) = \frac{1}{2}\{x(n) + x(-n)\} \\ x_o(n) = \frac{1}{2}\{x(n) - x(-n)\} \end{array}\right.$$

***

② Show that the product of two even signals or 2 odd signals is even signal, while the product of 1 odd & 1 even is a odd signal.

(A)

$x(n) = x_1(n) \cdot x_2(n)$ — ①

(i) Assume $x_1(n)$ & $x_2(n)$ are even signals.

$x_1(-n) = x_1(n)$ &

$x_2(-n) = x_2(n)$

To PROVE $x(n)$ is even

$$x(-n)=x(n)$$

replace n by -n. in ①

$$x(-n)= x_1(-n) \cdot x_2(-n)$$
$$x(-n) = x_1(n) \cdot x_2(n)$$
$$x(-n)= x(n)$$

HP

(ii) Assume $x_1(n)$ & $x_2(n)$ are odd signals.

$$x_1(n) = -x_1(n)$$
$$x_2(-n) = -x_2(n)$$

To prove $x(n)$ to be even

$$x(-n)= x(n)$$

replace n by -n in ①

$$x(-n)= x_1(-n) \cdot x_2(-n)$$
$$= x(-n) = -x_1(n) \cdot - x_2(n)$$
$$x(-n) = x_1(n) \cdot x_2(n)$$
$$x(-n)= x(n)$$

HP

(iii) Assume $x_1(n)$ is even signal & $x_2(n)$ is odd signal

$$x_1(-n)= x_1(n)$$
$$x_2(-n)= - x_2(n)$$

To prove $x(n)$ is odd signal

$$x(-n)=-x(n)$$

Replace n by -n in ①

$$x(-n)= x_1(-n) \cdot x_2(-n)$$
$$x(-n)= x_1(n) \cdot - x_2(n)$$
$$x(-n) = - x(n)$$

Hence The product of 1 odd & 1 even signal is odd signal.

MATLAB

(iv) DETERMINISTIC & RANDOM SIGNALS :-

The signals which can be represented / determined – whose occurrence is know. } —DETEMINSTIC
↓
By mathematical exp.

RANDOM SIGNALS – eg Noise of amplifier, Distortion @ Receiver, EEG
Occurrence of such signals is uncertain
↓
Brain Signals

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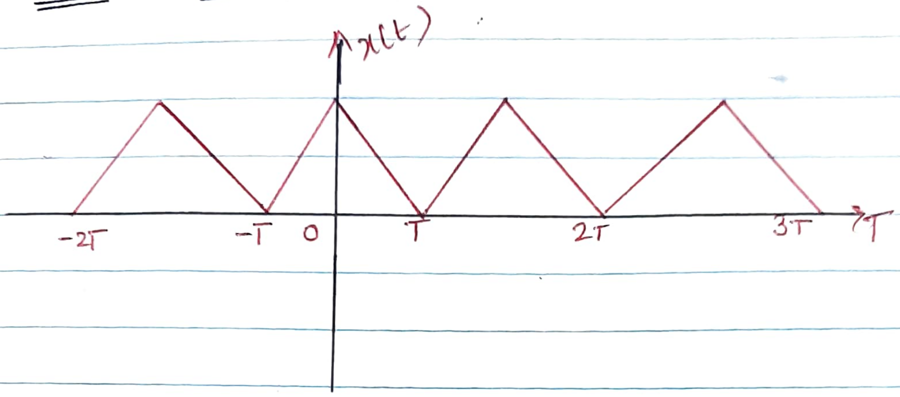
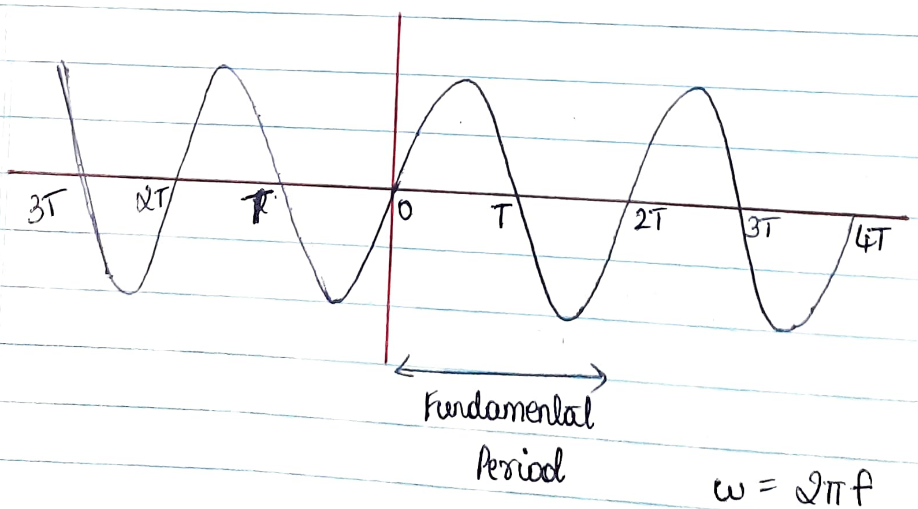
(V) PERIODIC & NON-PERIODIC / A-PERIODIC :-



A continuous time signal is said to be periodic iff

$$x(t) = x(t+T) \quad \forall t$$
$$= x(t+mT)$$
m = integer
±1, ±2, ±3 . . . .



$\omega = 2\pi f$

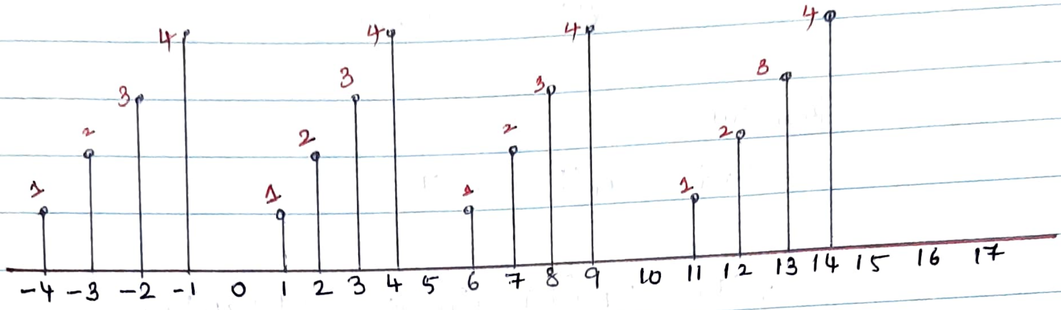
$\omega = \frac{2\pi}{T}$

DISCRETE PERIODIC SIGNAL

$$\Omega = \frac{2\pi}{N}$$

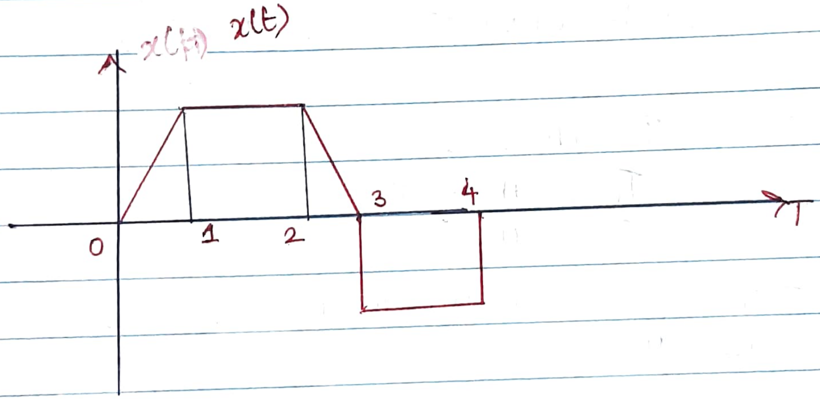
$$x(n) = x(n+N)$$
$$= x(n+mN)$$
$$m = \pm 1, \pm 2, \pm 3 \pm \ldots$$

ignal

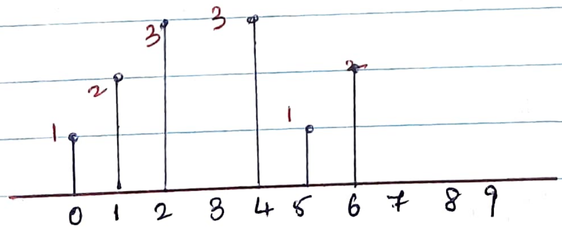


NON-PERIODIC :-

CONTINUOUS NON-PERIODIC :-



DISCRETE NON-PERIODIC :-



PERIODICITY OF SUM OF 2 SIGNALS :-

Let $x_1(t)$ is a periodic signal with fundamental period T_1
$x_2(t)$ is a periodic signal with fundamental period T_2

$x(t) = x_1(t) + x_2(t)$

Condition $\quad x(t)$ is periodic

$x_1(t) = x_1(t + mT_1)$ — ①

$x_2(t) = x_2(t + nT_2)$ — ②

Then sum

$x(t) = x_1(t) + x_2(t)$

$x(t) = x_1(t + mT_1) + x_2(t + nT_2)$

To say if Sum is Periodic - condition.

$x(t) = x(t+T)$

$x(t+T) = x_1(t + mT_1) + x_2(t + nT_2)$

$T = mT_1 = nT_2$

$$\frac{T_1}{T_2} = \frac{n}{m} = T$$

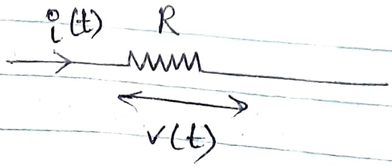
$T = \frac{T_1}{T_2} = \frac{n}{m} \Rightarrow$ Should Be a rational number.

The Sum of 2 periodic signals is periodic iff the ratio of their respective fundamental periods can be expressed as a rational number.

(ii) FUNDAMENTAL PERIOD:-

$T = LCM(T1, T2)$

(vi) ENERGY & POWER SIGNAL:-



continuous – integration
discrete – summation

$R = 1\Omega$ $\quad i(t) = x(t)$

$v(t) = i(t) \cdot R$

$v(t) = i(t) \cdot 1 = x(t)$

$$P(t) = \frac{v^2(t)}{R} = i^2(t) \cdot R$$

$$P(t) = v^2(t) = x^2(t) = i^2(t).$$

Total energy:

$$E = \int_{-\infty}^{\infty} x^2(t)\,dt$$

if $x(t)$ is complex

$$E = \int_{-\infty}^{\infty} |x(t)|^2\,dt$$

Average power:

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\,dt$$

if $x(t)$ is complex

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\,dt \approx \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

***

& if $x(t)$ is periodic its average power

$$P = \frac{1}{T} \int_{0}^{T} (x\,|t|)^2\,dt.$$

CONTINUOUS TIME SIGNALS.

DISCRETE TIME SIGNAL:-

Total energy, E

$$E = \sum_{n=-\infty}^{\infty} x(n)^2$$

POWER

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

2×5+1=11

-5 -4 -3 -2 -1 0 1 2 3 4 5

(2N+1)

if x(t) is periodic

average power:-

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2$$

*** A signal x(t) or x(n) is said to be an energy signal iff the Energy of the Signal is finite $0 < E < \infty$ & Power should be zero $P=0$

*** A signal x(t) or x(n) is said to be a power signal iff the power of the signal is finite $0 < P < \infty$ & Energy should be ∞, $E = \infty$

****** Usually Periodic Signals & Random Signals are Power Signals. Aperiodic & Deterministic Signals are Energy Signals.

ax+b

BASIC. OPERATIONS OF SIGNALS:-

Dependent variable

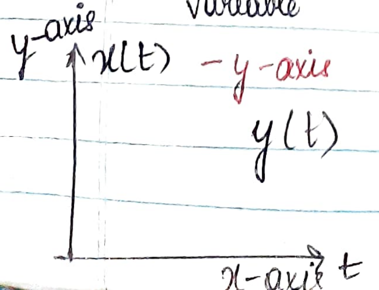
x(t) - y-axis

y(t)

independent variable

- x-axis.

t



AMPLITUDE SCALING :-

① $x(t) =$

$$\boxed{y(t) = c \cdot x(t)}$$

② ADDITION :-

$x_1(t)$ & $x_2(t)$

$$y(t) = x_1(t) + x_2(t)$$

$$\boxed{y(t) = x_1(t) + x_2(t)}$$

③ MULTIPLICATION :-

$$\boxed{y(t) = x_1(t) \cdot x_2(t)}$$

Eg- modulation Technique
– AM/FM.

④ DIFFERENTIATION → physical device performs differentiation – INDUCTOR

$$\boxed{y(t) = \frac{d}{dt} x(t)}$$

⑤ INTEGRATOR physical device performing integration opⁿ – CAPACITOR.

$$y(t) = \int_{-\infty}^{\infty} x(t)\, dt$$

INDEPENDENT VARIABLES :-

① TIME SCALING
② TIME SHIFTING $x_1(t - t_0)$
③ REFLECTION OR TIME REVERSAL OR FOLDING. $x(-t)$

Eg $x(t) = x_1(at)$

$$y(t) = x(\underbrace{-at - t_0})$$

↳ within the Brackets comes in the x-axis.

(1) TIME SHIFTING :-

Let $x(t) \Rightarrow$ CTS

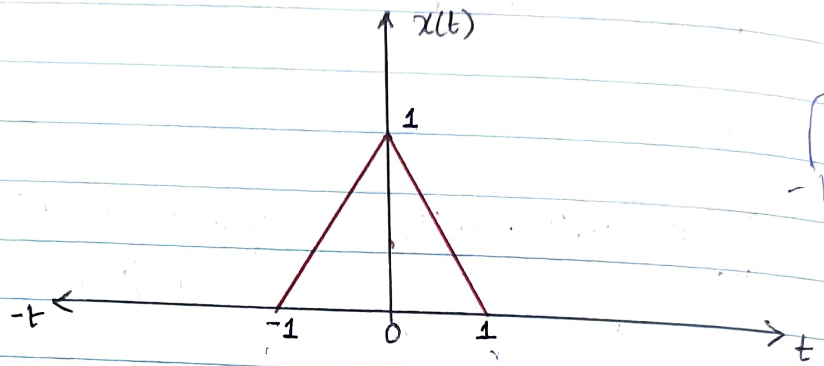
$$y(t) = x(t - t_0)$$

or

$$y(t) = x(t - a)$$

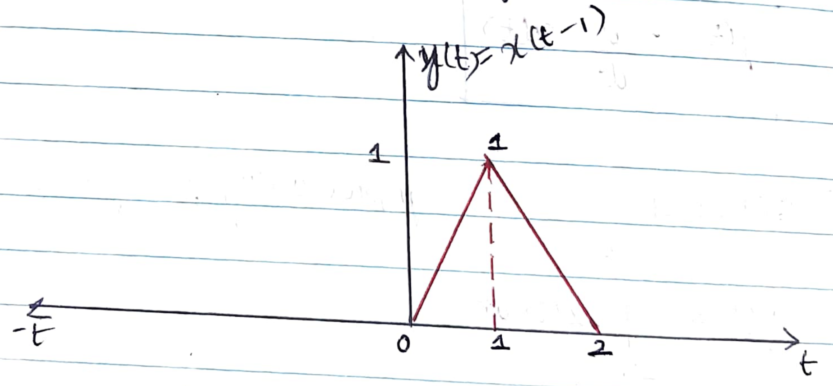
*** If $a > 0$, for positive values of a → RIGHT SHIFT

*** If $a < 0$, for negative values of a → LEFT SHIFT



$$y(t) = x(t-1)$$

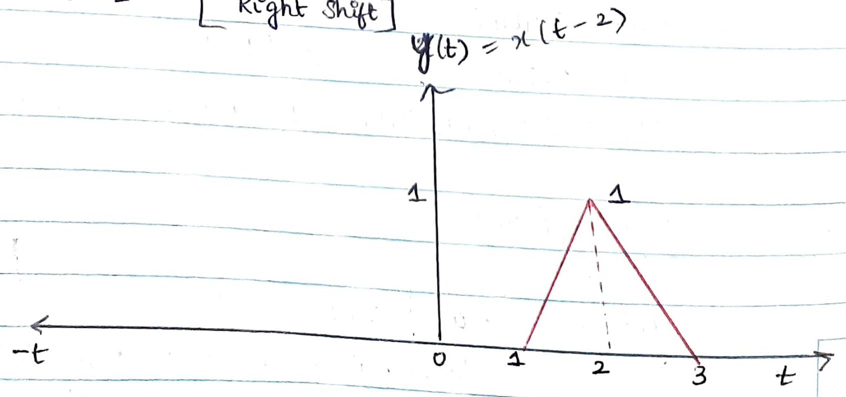
$a = 1$ $(a > 0)$ Right shift



amplitude remains the same.

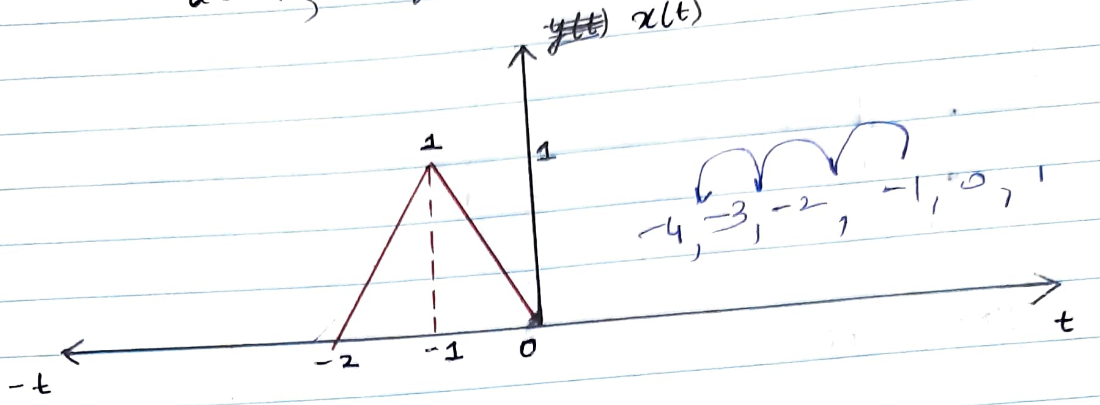
$$y(t) = x(t-2)$$

$a = 2$ [Right shift]



Let $y(t) = x(t+1) = x(t-(-1))$

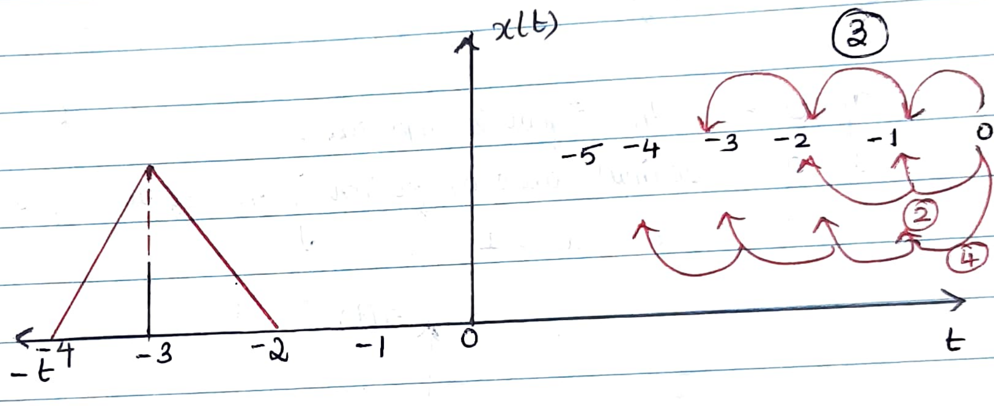
$a = -1$, Left shift operation.



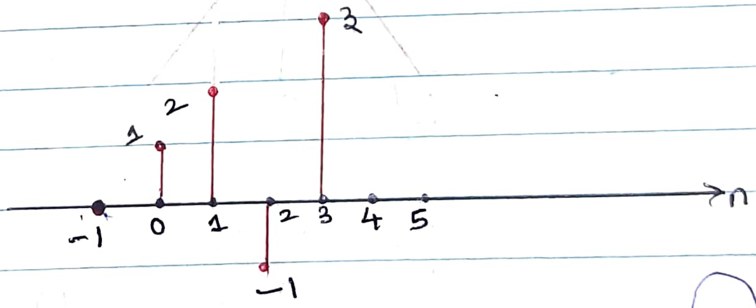
$y(t) = x(t+3) = x(t-(-3))$

$a = -3$, — Left shift operation

$a < 0$

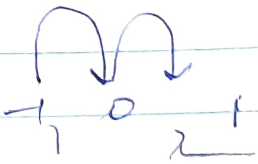


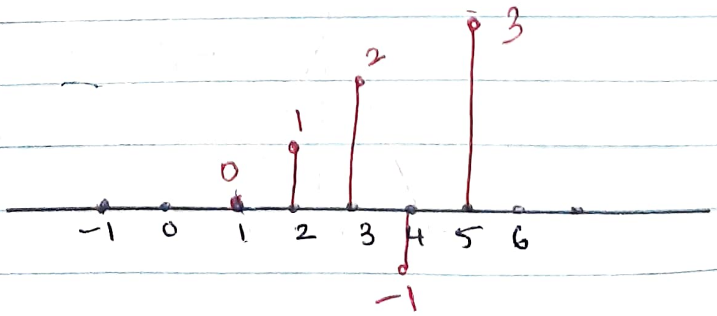
FOR DISCRETE TIME SIGNALS :-



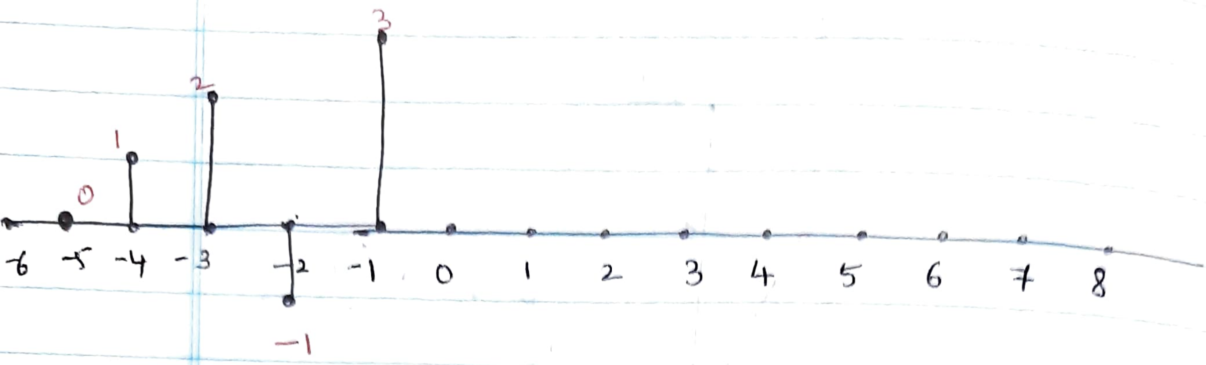
$y(n) = x(n-2)$

$a = 2$ Right shift





$y_2(n) = x(n+4) = x(n-(-4))$



(2) TIME SCALING :-

affects x-axis

$x(t) \Rightarrow$ CTS

$y(t) = x(at)$ — Time scaling

a = scaling factor.

$\boxed{y(t) = a\,x(t)}$

— amplitude scaling

→ affects y-axis.

If $a > 0$ the signal is compressed.

If a is decimal value or fraction } the signal is (fraction) is expanded by ½ (0.5)

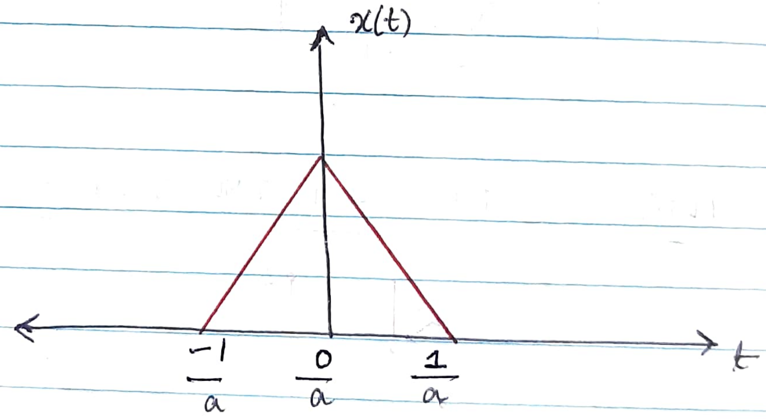
$0 > a > 1$

$\boxed{0 < a < 1}$

$y(t) = x(at)$

α

$a > 0$



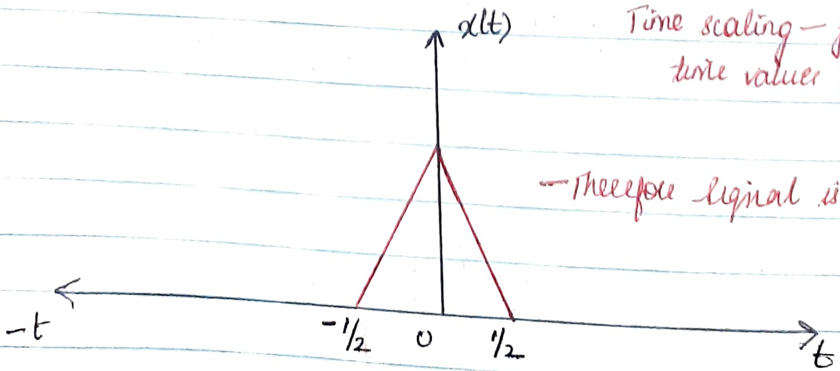
$y(t) = x(2t)$

$a = 2$

Hint :- For Continuous time Signals Time scaling — just divide the time values by a.

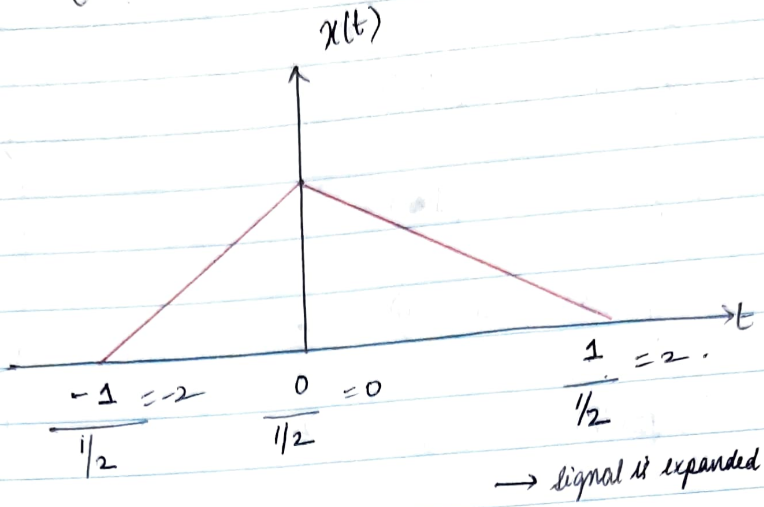
— Therefore signal is compressed.

$y(t) = x(2t)$



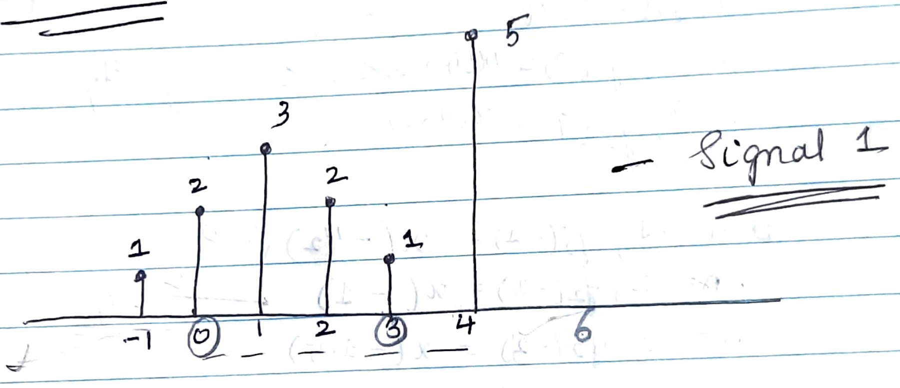
$y_2(t) = x(0.5t)$

$= x\left(\frac{1}{2}t\right)$



→ signal is expanded as it is CTS.

DISCRETE TIME SIGNALS :-



$x(-3)$

$$\boxed{y(n) = x(3n)}$$

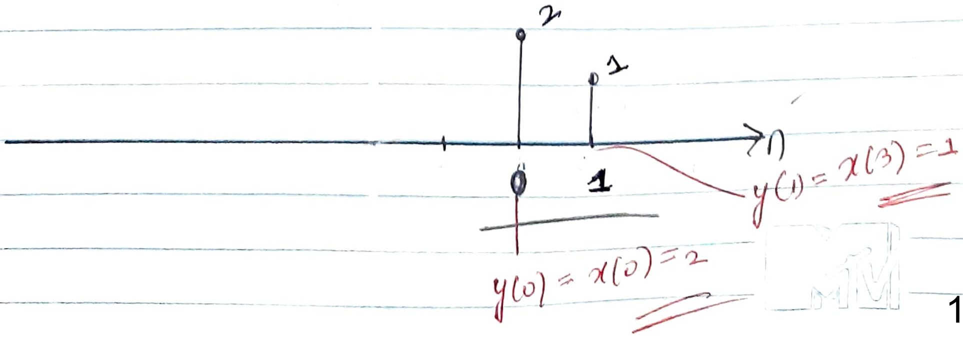
$n=0$, $y(0) = x(0)$ ✓

$n=1$, $y(1) = x(3)$ ✓

$n=2$, $y(2) = x(6)$ —— stop. $\propto$

$n=-1$, $y(-1) = x(-3)$ —— stop.

$x--2$



② $y_2(n) = x(0.5n)$

$= x(\frac{1}{2}n)$

$n=0, \quad y_2(0) = x(0) = 2$

$n=1, \quad y_2(1) = x(1/2) = 0$

$n=2, \quad y_2(2) = x(1) = 3$

$n=3, \quad y_2(3) = x(1.5) = 0$

$n=4, \quad y_2(4) = x(2) = 2$

$n=5, \quad y_2(5) = x(2.5) = 0$

$n=6, \quad y_2(6) = x(3) = 1$

$n=7, \quad y_2(7) = x(3.5) = 0$

$n=8, \quad y_2(8) = x(4) = 5 \longrightarrow$ Stop.

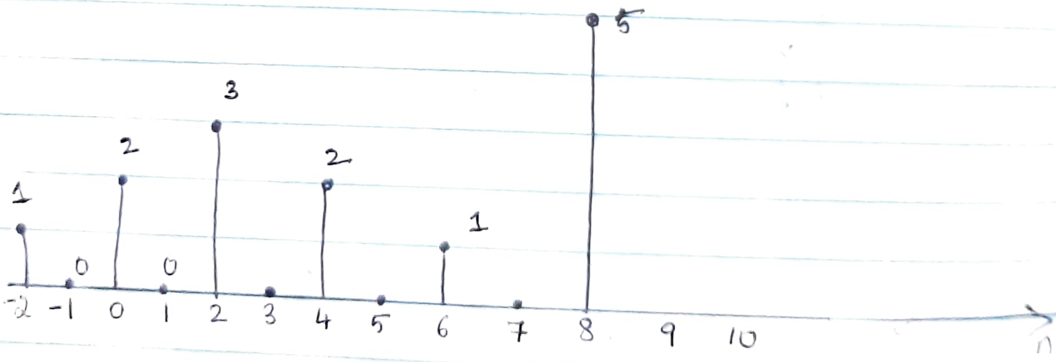
$n=9, \quad y_2(9) = x(4.5)$

at $n=-1, \quad y_2(-1) = x(-1/2) = 0$

$n=-2, \quad y_2(-2) = x(-1) \longrightarrow 1$ Stop

$n=-3, \quad y_2(-3) = x(-1.5)$

Refer Signal [illegible]



– Signal is expanded.

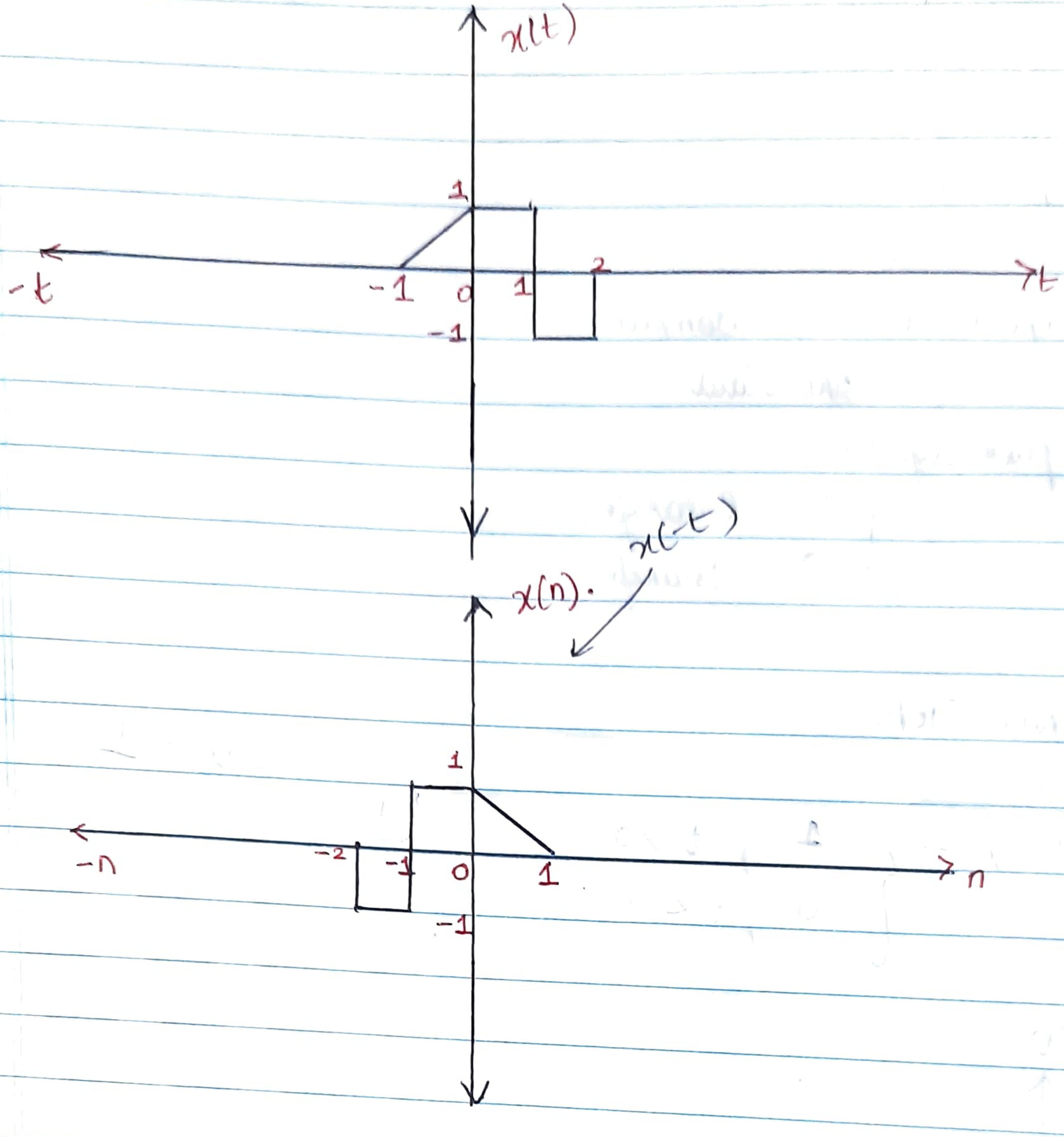
③ REFLECTION/TIME REVERSAL/FOLDING:

$x(t) \rightarrow$ CTS.

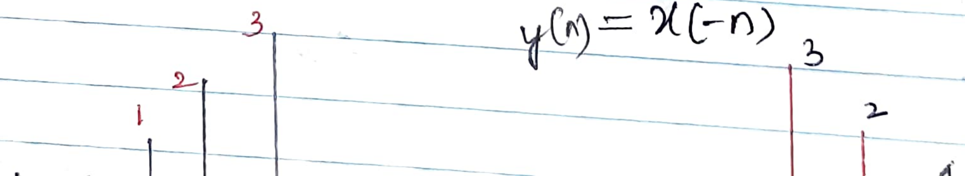
$x(n) =$ DTS.

$y(t) = x(-t).$

$y(n) = x(-n)$ } — Reflection



DISCRETE TIME SIGNALS :



19/02/18

BASIC CONTINUOUS SIGNALS:-

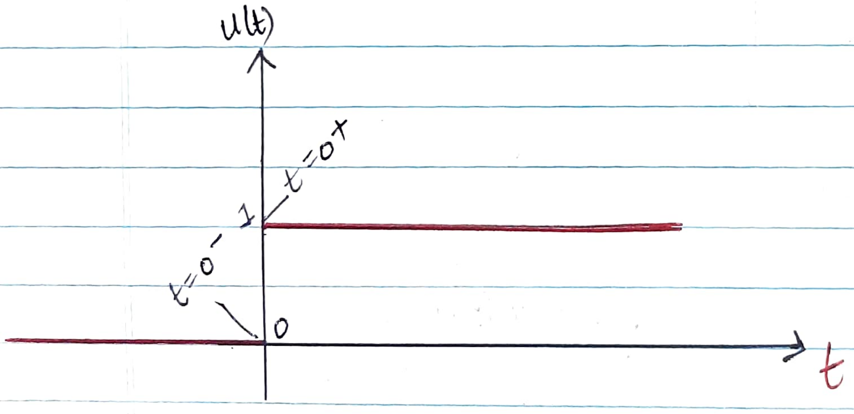
1) unit step
2) Unit impulse
3) unit ramp
4) exponential
5) Sinusoidal
6) exponential damped sinusoidal
7) pulse signals
   - → Rectangle
   - → Triangle

UNIT STEP.

CONTINUOUS

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \quad ①$$

$u(t-t_0) = 1, t$



$$u(t) = \begin{cases} 1, & t \geq 0^+ \\ 0, & t \leq 0^- \end{cases}.$$

Replace t By t-to

$$u(t-t_0) = \begin{cases} 1, & t-t_0 > 0 \\ 0, & t-t_0 < 0 \end{cases}.$$

OR

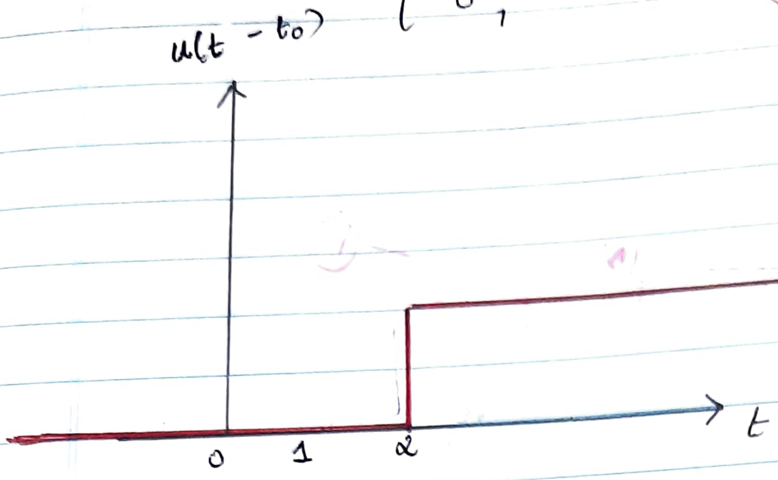
$\{1, t \geq 0 ; 0, t < 0\}$

$$u(t-t_0) = \begin{cases} 1, & t > t_0 \\ 0, & t < t_0 \end{cases}$$

$$u(t+t_0) = \begin{cases} 1, & t > -t_0 \\ 0, & t < -t_0 . \end{cases}$$

Assume

$t_0 = 2$



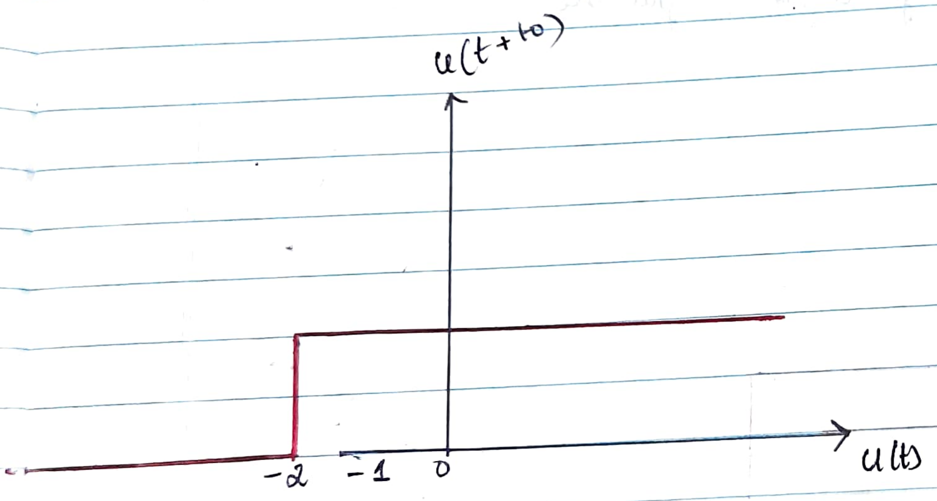
$-t_0 > 0$

$t > t_0$

Assume

$t_0 = 2$

$u(t \oplus t_0)$



Replace t By -t

$$u(-t) = \begin{cases} 1, & -t > 0 \\ 0, & -t < 0 \end{cases}$$

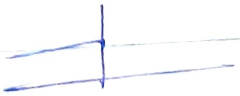
$u(-t) = 1, -t > 0$

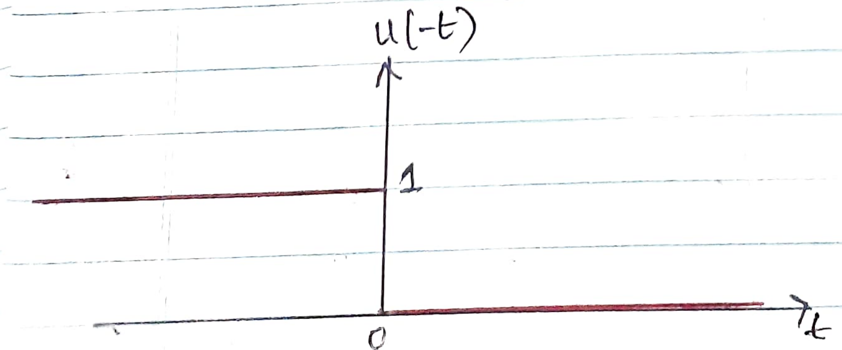
$t < 0$

$0, t > 0$

or

$$u(-t) = \left\{ \begin{matrix} 1, & t < 0 \\ 0, & t > 0 \end{matrix} \right\}.$$

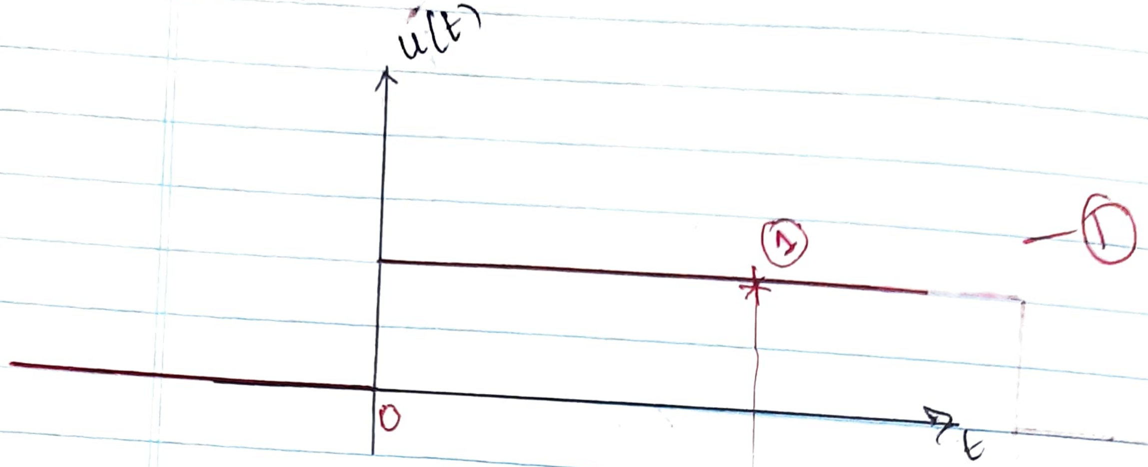




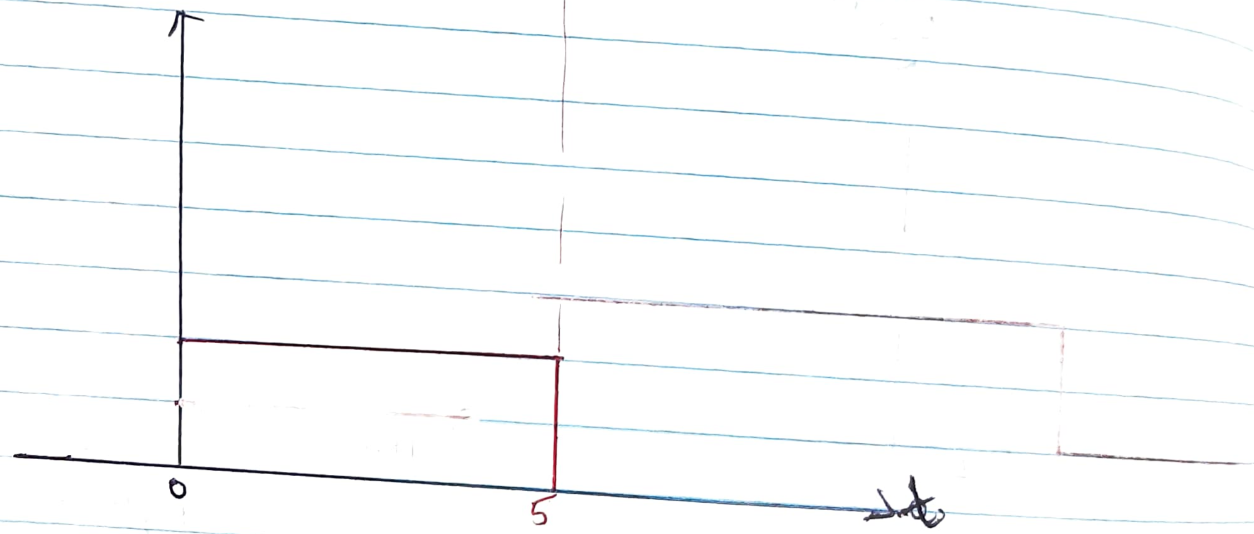
DISCONTINOUS FUNCTION :-

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

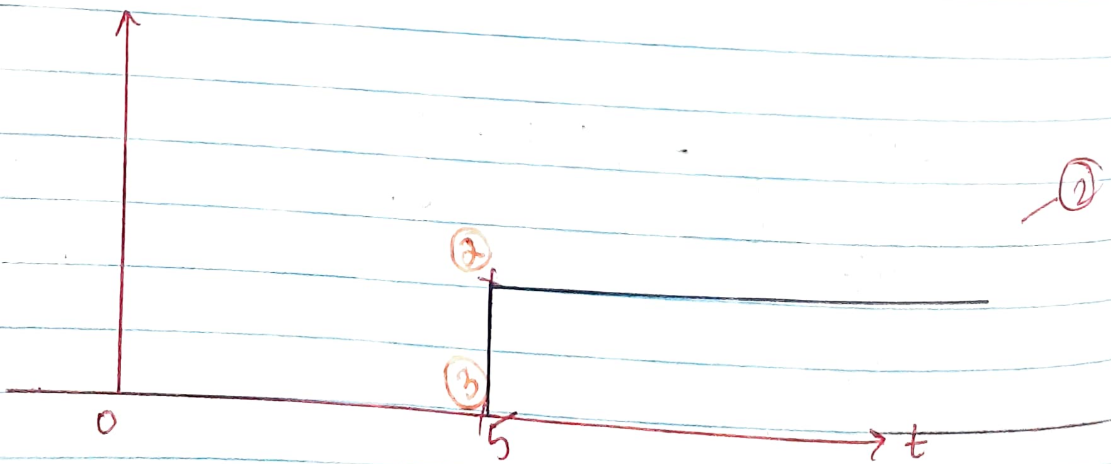
u(t) =



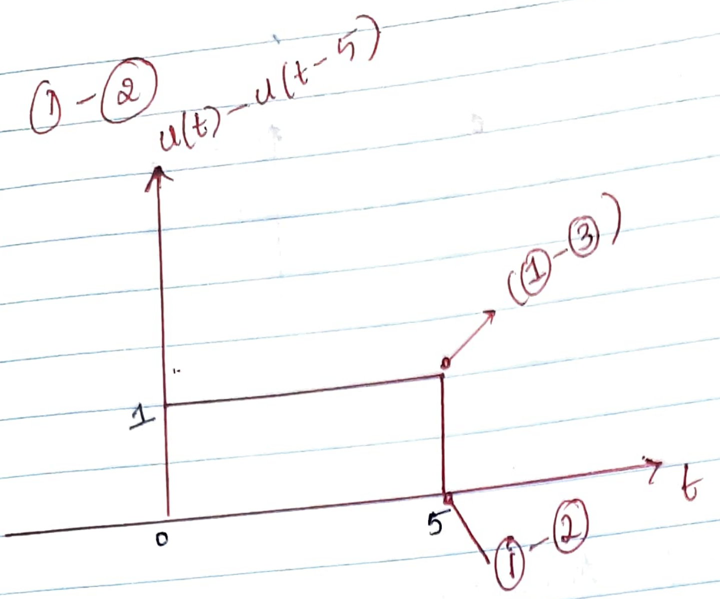
If u wanna get a discrete signal like



∴ u(t-5)



$u(t)-$ to.



$\delta(t)$.

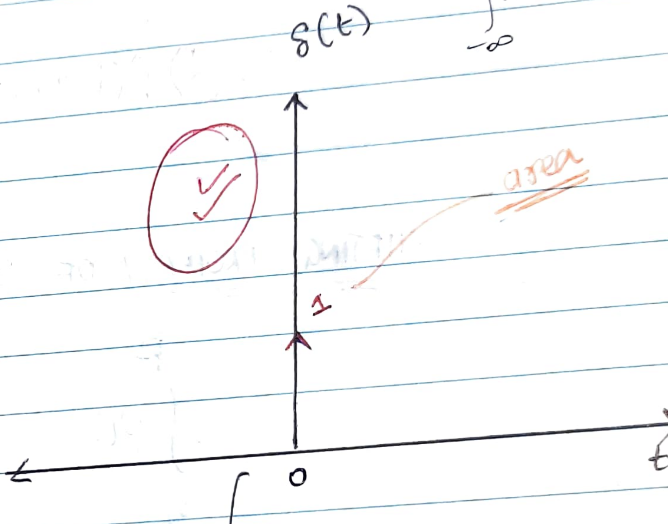
② UNIT IMPULSE FUNCTION :-

$$\delta(t) = \begin{cases} 1, & t=0 \\ 0, & t\neq 0 \end{cases}$$

$\delta(t) = \int_{-\infty} \delta(t)\,dt = 1$

$$\delta(t)=0 \qquad t\neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

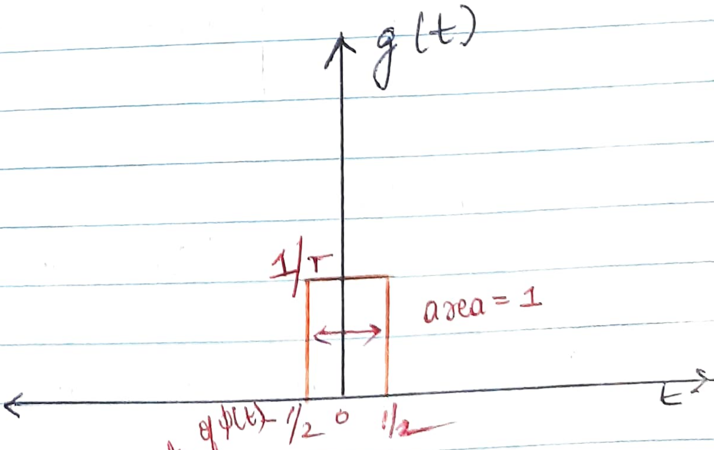


$\delta(t)=\delta(-t)$ — EVEN FNC

width = small
length = infinitely large
area = 1

$$\boxed{\delta(t) = \lim_{T\to 0} g(t)}$$

OR

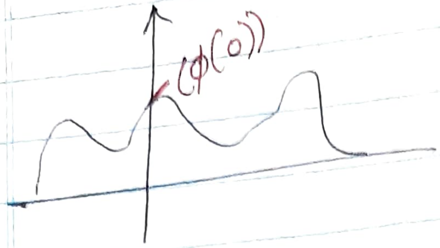


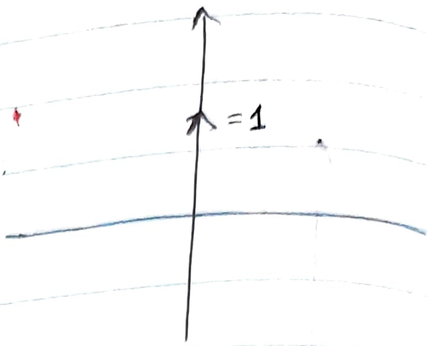
PROPERTIES OF IMPULSE FUNCTION :-

amplitude of φ(t) at t=0.

① PRODUCT PROPERTY :-

$$\phi(t)\cdot\delta(t) = \phi(0)\cdot\delta(t)$$

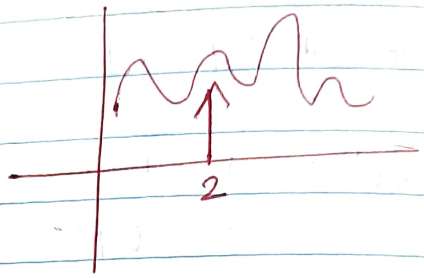




$$\phi(t)\cdot\underbrace{\delta(t-t_0)} = \underline{\phi(t_0)}\cdot\underline{\delta(t-t_0)}$$

Eg:- $\phi(t)\cdot\delta(t-2)$

$=\phi(2)\,\delta(t-2)$.



② SHIFTING PROPERTY OF IMPULSE :-

$$\int_{t_1}^{t_2}\phi(t)\cdot\delta(t-t_0)\,dt = \begin{cases}\phi(t_0), & t_1 < t_0 < t_2\\ 0, & \text{else where.}\end{cases}$$

③ TIME SCALING PROPERTY :-

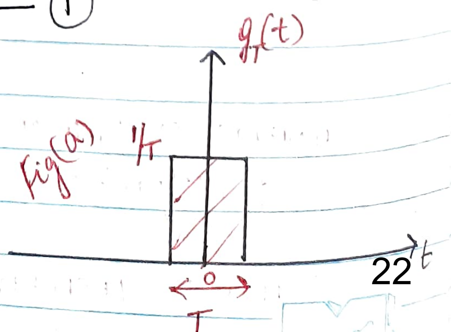
$$\boxed{\delta(at) = \frac{1}{a}\,\delta(t), \quad a>0}$$

$\lim_{T\to 0} g_T(t)$

PROOF :-

$$\delta(t) = \lim_{T\to 0} g_T(t) \quad — ①$$

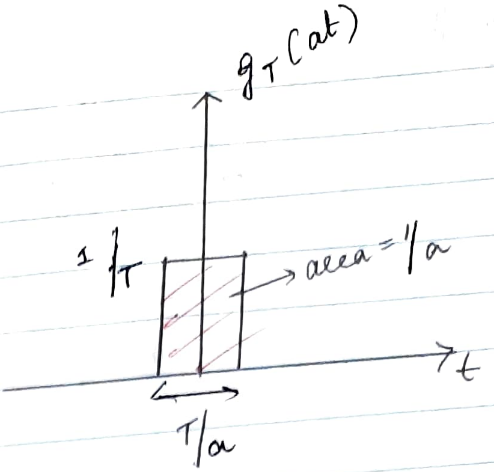
Substitute ~~t~~ by at in ①



$$\delta(at) = \lim_{T\to 0} g_T(at) \quad — ②$$

From fig b & c

Fig(b)



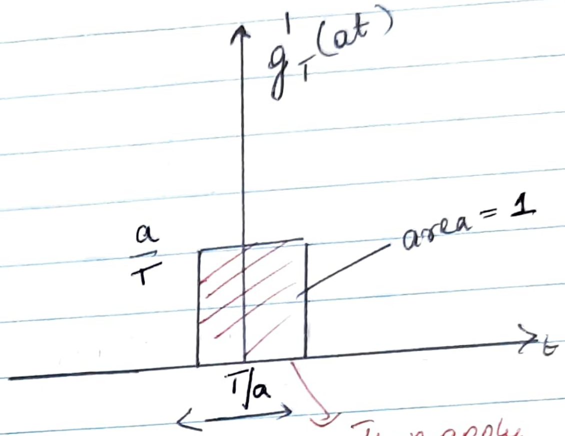
$$g_T'(at) = a\; g_T(at)$$

$$g_T(at) = \frac{1}{a}\; g_T'(at) \quad -③$$

Substitute ③ in ②

$$\boxed{\delta(at) = \frac{1}{a}\left(\lim_{T\to 0} g_T'(at)\right)}$$

Fig(c)



$$\boxed{\delta(at) = \frac{1}{a}\;\delta(t)} \quad — X \cdot X \cdot X$$

If we apply limit it will be

which is impulse fnc.

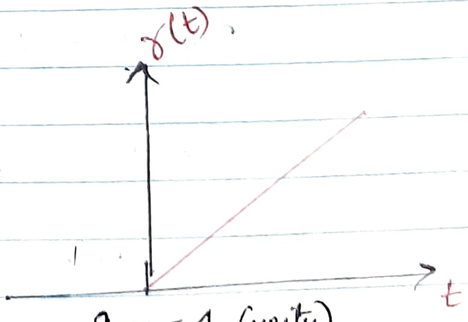
NOTE:- In a unit impulse function, is a derivative of the step function,

$$\cancel{\frac{d}{dt}\{u(t)\} = 1}$$

u(t) · t

4. RAMP FUNCTION :-

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



Slope = 1 (unity)

$r(t) = t.\underbrace{u(t)}_{\text{unit step}}$

$$\boxed{r(t) = t \cdot u(t)}$$

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$x(t) = Be^{at}$

④ EXPONENTIAL SIGNALS :-

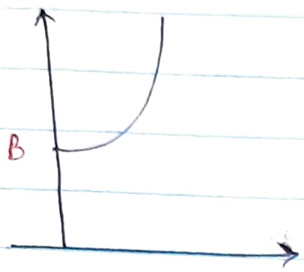
CONTINUOUS FNC/s :

$$x(t) = Be^{at}$$

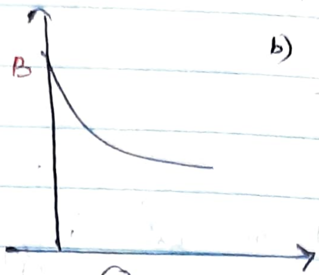
B - amplitude @ $t = 0$

a) $a > 0$ - growing exponential.

b) $a < 0$ - decaying exponential.



ⓐ



ⓑ

⑤ SINUSOIDAL SIGNALS :-

sine / cosine fnc.

$$x(t) = \underbrace{A}_{\text{amplitude}} \cos(\underbrace{\omega}_{\text{angular freq}} t + \underbrace{\phi}_{\text{phase}})$$

Be^{at}

$$\omega = 2\pi f$$

$$T = \frac{2\pi}{\omega}$$

$$\boxed{\omega T = 2\pi}$$

$\omega T = 2\pi$

$$x(t) = x(t+T) = x(t+nT)$$

$$x(t+T) = A\cos[\omega(t+T) + \phi]$$

$$x(t+T) = A\cos(\omega t + \omega T + \phi)$$

$$x(t+T) = A\cos(\omega t + 2\pi + \phi)$$

$$x(t+T) = A\cos(\omega t + \phi) = x(t)$$

$\therefore x(t) = x(t+T)$ ⟶ Hence it is periodic

$A\ \sin$ $\quad A\ e^{-\alpha t}\ \sin$ $\quad A\sin$

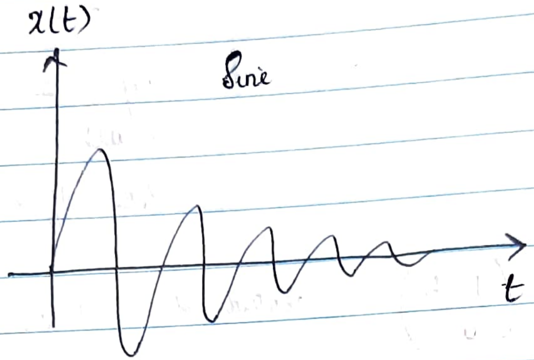
⑥ exponentially damped Sinusoidal Signals :- — ON Multiplying ④ & ⑤.

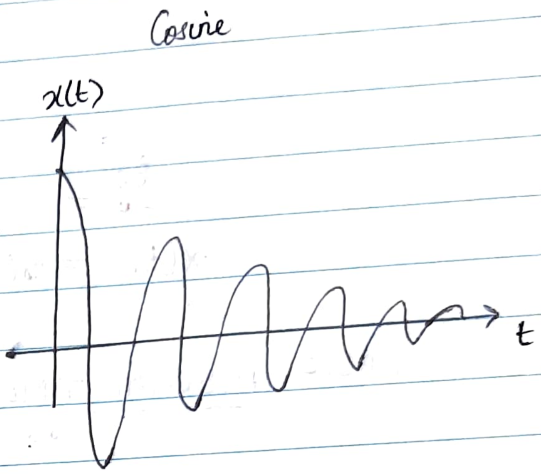
~~$x(t) = A e^{\alpha t}$~~

$$x(t) = A\ e^{-\alpha t}\ \sin\ (\omega t + \phi)$$ ✓

$\forall \quad \alpha > 0$

onential.
nential.





$|t| < 0.5$

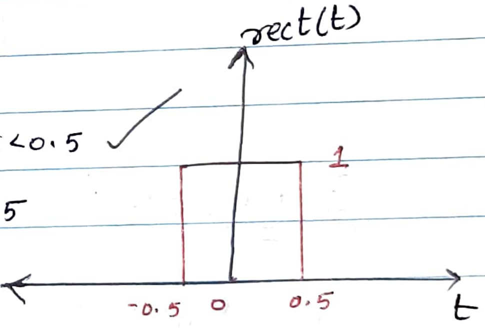
⑦ PULSE SIGNALS :-

Be^{at}

(i) RECTANGULAR PULSE :-

$$\text{rect}(t) = \begin{cases} 1, & -0.5 < t < 0.5 \\ & \text{or} \\ & |t| < 0.5 \\ 0, & \text{elsewhere} \end{cases}$$ ✓

$\text{rect}(t) = \{ 1, |t| < 0.5$



b=0
a=1

NOTE :- b indicates centre point, a indicates width.

$\text{rect}(t) = \{ 1$

If $x(t) = \text{rect}\left(\dfrac{t-b}{a}\right)$ centred @ $t=b$ width $= a$. } Describes a rectangular pulse of width a centred at $t=b$.

Here $a=1$
$b=0$.

$\text{rect}\left(\dfrac{t-b}{a}\right)$

$a = w$

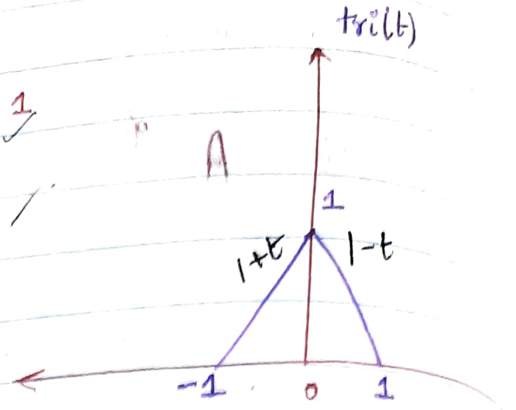
$x(t) = \text{rect}\left(\dfrac{t-b}{a}\right)$

ba

(ii) TRIANGULAR PULSE:-

$$Tri(t) = \begin{cases} 1-|t|, & |t| < 1 \\ 0, & \text{elsewhere} \end{cases}$$

Λ



Height = 1
width = 2
Area = $\frac{1}{2}$ bh = 1

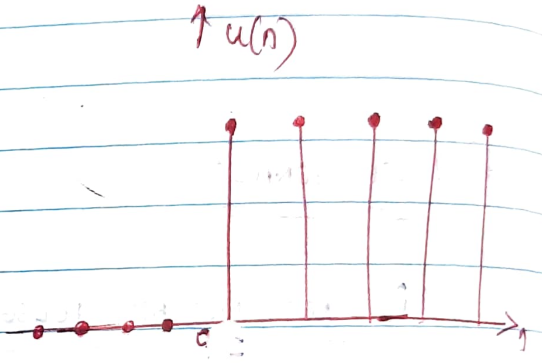
Eg:-

$$x(t) = tri\left(\frac{t-b}{a}\right) = \text{centred at } t=b \text{ \& width} = \boxed{2a}$$

BASIC DISCRETE TIME SIGNALS :-

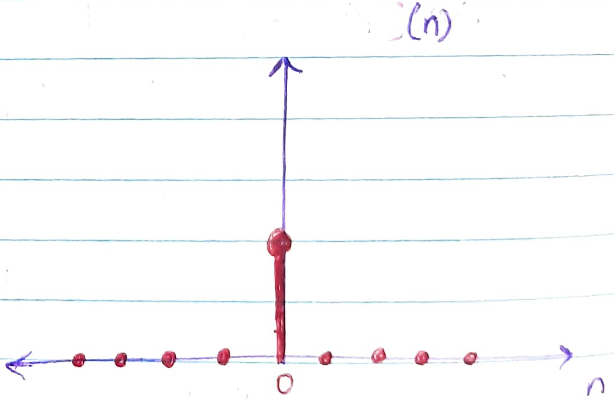
① Unit step :-

$$x(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



② IMPULSE :-

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



$x(t)\cdot\delta(t-t_0) = x(t_0)$

PROPERTIES OF IMPULSE SIGNALS:-

(i) PRODUCT PROPERTY:-

$$x(n)\cdot\delta(n-k) = x(k)\cdot\delta(n-k)$$

$1-|t|, |t|<1$ $\quad 1+t \quad 1-t \quad 1-|t|.$

$|t| \geq 1.$

② SHIFTING PROPERTY :-

$$\sum_{n=N_1}^{N_2} \underbrace{x(n)\cdot\delta(n-k)} = \begin{cases} x(k) & , n_1 \leq k \leq n_2 \\ 0, & \text{elsewhere} \end{cases}$$

(or)

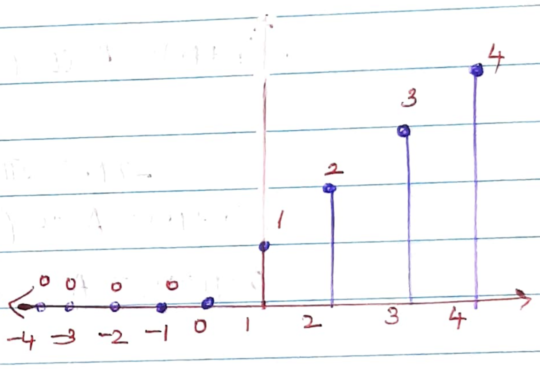
$$\sum_{n=-\infty}^{\infty} x(n)\cdot\delta(n-k) = x(k).$$

$r(t) = t\,u(t)$

③ RAMP FUNCTION | SIGNAL :-

$$r(n) = \begin{cases} n, & n \geq 0 \\ 0, & n<0 \end{cases}$$

$$\boxed{r(n) = n\cdot u(n)}$$ ✓



④ EXPONENTIAL FUNCTIONS :-

NOTE

$x(t) = B\,e^{at} = B\,e^{\alpha t}$ $\qquad x(n) =$

we get discrete signal by sampling continuous signals.

$t \to nT$

$x(nT) = B\left(e^{\alpha nT}\right)$

$e^{\alpha T} = \gamma$

$$\boxed{x(n) = B\gamma^{n}}$$

$x(t) = Be^{at}$

$x(nT) = Be^{anT}$

$B(e^{aT})^{n}$

$B\gamma^{n}$

when $0 < \gamma < 1$, we have decaying exponential

$\gamma > 1$, " " growing "

$$x(n+N) = A\cos(\Omega(n+N)+\phi)$$

$$\frac{1}{N} = \frac{\Omega}{2\pi} = \frac{f}{q}$$

$$x(n+N) = A\cos(\Omega n+\Omega N+\phi)$$

N = q samples

$$\Omega N = 2\pi$$

$$x(n+N) = A\cos(\Omega n+2\pi+\phi)$$

$$x(n+N) = A\cos(\Omega n+\phi)$$

$$\Omega = \frac{2\pi}{N}$$

$$N = \frac{2\pi}{\Omega}$$

expressed in fraction.

$$\boxed{\frac{\Omega}{2\pi} = \frac{1}{N} = \frac{f}{q}} =$$

N = q samples.

EXPONENTIALLY DAMPED SINUSOIDAL SIGNALS:-

$$x(n) = B\gamma^{n}\sin(\Omega n+\phi)$$

$0 < \gamma < 1$ — decaying exponential

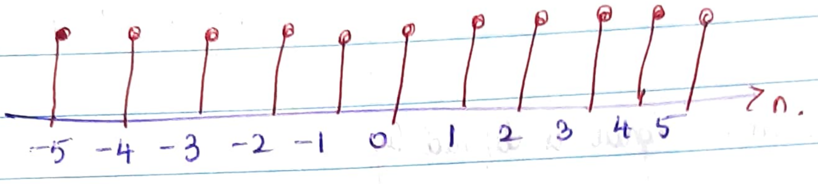
(7.) PULSE SIGNALS :-

−N . . . −1 0 1

(1.) Rectangular Pulse :-

$$\text{rect}\left(\frac{n}{2N}\right) = \begin{cases} 1, & |n| \leq N \\ 0, & \text{elsewhere} \end{cases}$$

eg $N = 5$
$-N = -5$



— (11) → (2N+1)

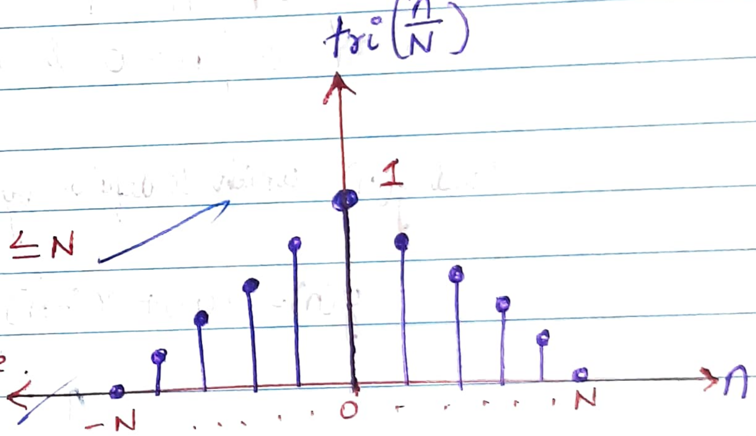
width = $\boxed{2N+1}$ over the range $[-N, N]$

rect(t)

(2.) Triangular pulse

rec$\left(\frac{n}{2N}\right)$

$$\text{tri}\left(\frac{n}{N}\right) = \begin{cases} 1 - \frac{|n|}{N}, & |n| \leq N \\ 0, & \text{elsewhere.} \end{cases}$$



$1 - |t|, \ |t| < 1$

when $n = \pm N$

if $N = 0$, $1 - 0$ (Maximum)

$1 - \frac{|n|}{N}$

No. of Sample → $\boxed{2N+1}$

$x(-at-b)$

$x(at-b)$.

NOTE :-

① $x(-at-b)$.

— PRECEDENCE :-

— ① shifting
② Scaling
③ Reflection

Determine whether the foll signals are energy signals, power signals or neither.

$x(t) = e^{-at} u(t), \quad a>0$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |e^{-at} u(t)|^2 dt$$

$$= \int_{0}^{\infty} e^{-2at} . 1 \, dt = \int_{0}^{\infty} e^{-2at} dt$$

$$= \left. \frac{e^{-2at}}{-2a} \right]_0^{\infty} = \frac{e^{-2a\infty} - e^{-20}}{-2a}$$

$$= - \frac{[-e^0]}{2a} = \frac{1}{2a} \; ; \quad 0 < E < \infty$$

power =

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{0}^{T} e^{-2at} dt$$

$$= \lim_{T \to \infty} \frac{1}{2T} \left[\frac{e^{-2at}}{-2a} \right]_0^T$$

$$= \lim_{T \to \infty} \frac{1}{2T} \left[\frac{1 - e^{-2aT}}{+2a} \right]$$

$$= \underline{\underline{0}}$$

$0 < E < \infty$

& $p = 0$

∴ energy signal

② $x(t) = u(t)$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} (u(t))^2 dt$$

$$= \int_0^1 1\, dt = t\Big]_0^{\infty} = \infty - 0 = \underline{\underline{\infty}}$$

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_0^T 1\, dt$$

$$= \lim_{T \to \infty} \frac{1}{2T} [t]_0^T$$

$$= \lim_{T \to \infty} \frac{1}{2T} \times T = \underline{\underline{\frac{1}{2}}}$$

$E = \infty$, & $0 < p < \infty$ $\therefore$ $x(t)$ is a power signal

③ $x(t) = t \cdot u(t)$

$$x(t) = \begin{cases} t. & \text{for } t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{0}^{\infty} t^2 dt$$

$$= \frac{t^3}{3}\Big]_0^{\infty} = \underline{\underline{\infty}}$$

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{0}^{T} t^2 dt$$

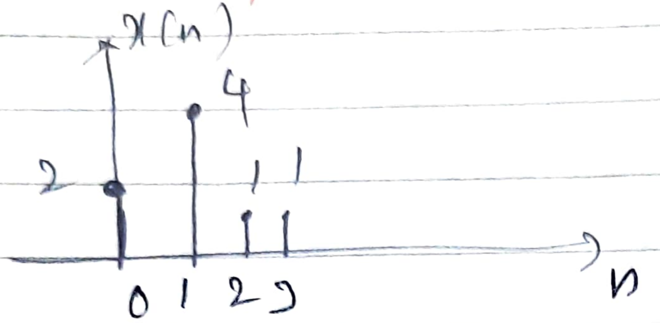
$$= \lim_{T \to \infty} \frac{1}{2T} \times \frac{t^3}{3}\Big]_0^{T}$$

$$= \lim_{T \to \infty} \frac{1}{2T} \times \frac{T^3}{3} = \lim_{T \to \infty} \frac{T^2}{6} = \underline{\underline{\infty}}$$

$E = \infty$, $P = \infty$

∴ $x(t)$ is neither a power signal nor a energy signal

④ $x(n) = \{\underset{\uparrow}{2}, 4, 1, 1\}$



$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=0}^{3} |x(n)|^2 = x(0)^2 + x(1)^2 + x(2)^2 + x(3)^2$$

$$= 2^2 + 4^2 + 1^2 + 1^2 = \underline{\underline{22J}}$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} (22)$$

$$= \frac{1}{\infty}(22) = 0$$

$E < \infty$

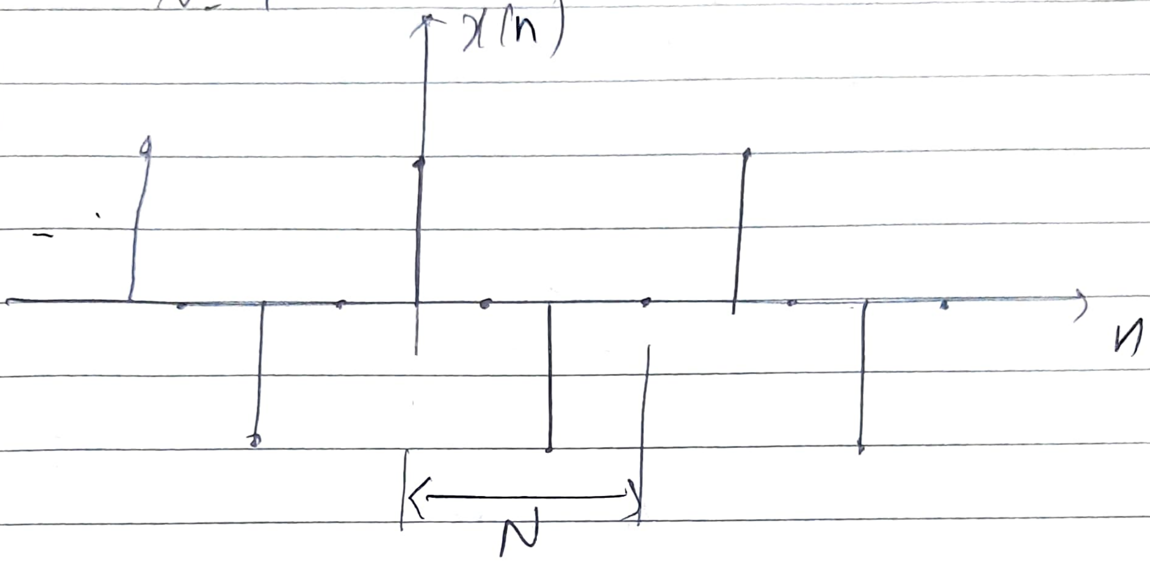
$P = 0$

∴ energy signal

(57) $x(n) = \cos\left(\frac{n\pi}{2}\right)$

$$x(n) = \{0, -1, 0, \underbrace{\underset{\uparrow}{1}, 0, -1, 0}_{N}, 1 \ldots\}$$

$N = 4$



x(n) is periodic signal with $N = 4$

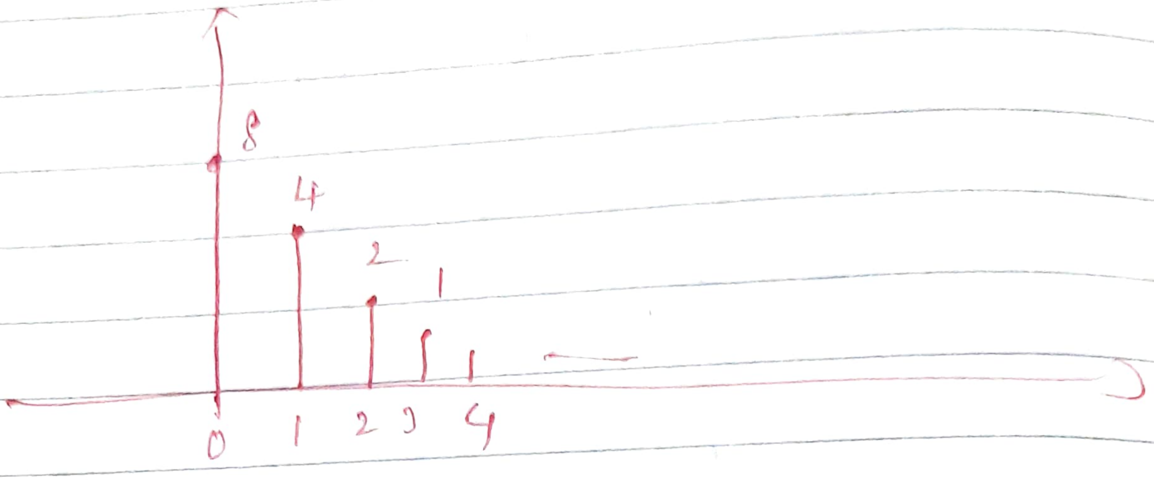
$$\therefore P = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2$$

$$= \frac{1}{4} \sum_{n=0}^{3} |x(n)|^2$$

$$= \frac{1}{4}[1+1] = \underline{\underline{0.5\,W}}$$

periodic signals are power signals

⑥ $x(n) = 8\,(0.5)^n u(n)$



$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$(0.5)^{2n}$
$=(0.5^2)^n$
$=(0.25)^n$

$$= \sum_{n=0}^{\infty} 64\,(0.25)^n$$

$$\boxed{\text{WKT} \quad \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad |a|<1}$$

$$= 64 \sum_{n=0}^{\infty} (0.25)^n$$

$$= 64 \times \frac{1}{1-0.25}$$

$$= \underline{85.33\,J}$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} (85.33) = \underline{\underline{0}}$$

∴ energy signal

⑦ $x(t) = A e^{jw_0 t} \quad -\infty < t < \infty$

$$|x(t)| = |A e^{jw_0 t}| = A$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 = \int_{-\infty}^{\infty} A^2 dt = A^2 t \Big]_{-\infty}^{\infty} = \underline{\underline{\infty}}$$

$$P = \underset{T \to \infty}{Lt} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$= \underset{T \to \infty}{Lt} \frac{1}{2T} \int_{-T}^{T} (A)^2 dt$$

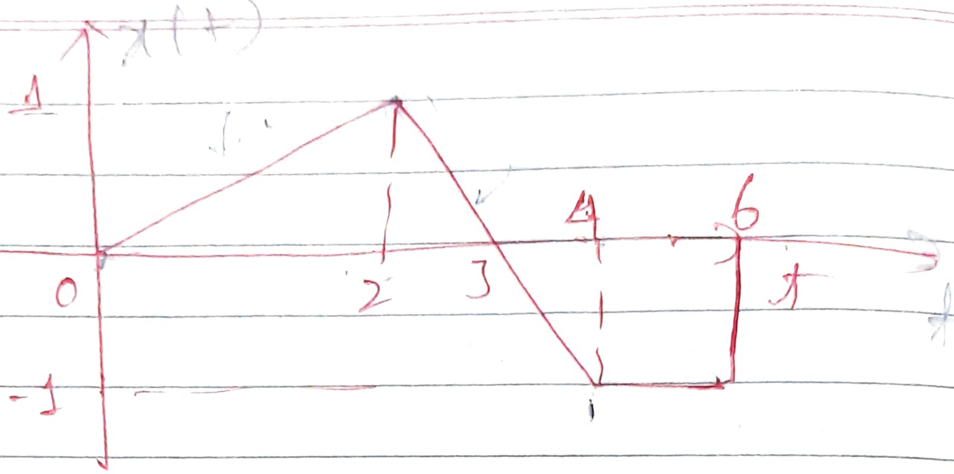
$$= \underset{T \to \infty}{Lt} \frac{1}{2T} \left[A^2 t\right]_{-T}^{T}$$

∴ power signal

$$= \underset{T \to \infty}{Lt} \frac{1}{2T} \times 2A^2 T = \underline{\underline{A^2}}$$

⑧

②

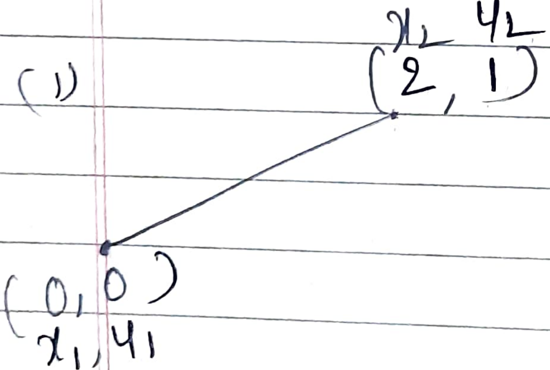


$$\frac{y_2-y_1}{x_2-x_1} = \frac{y-y_1}{x-x_1}$$

$y = x(t)$

$x = t$

(1)



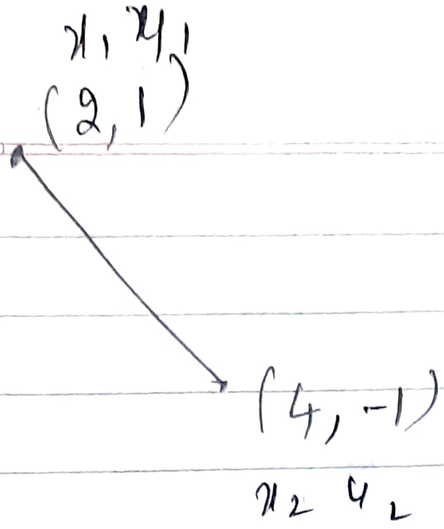
$$\frac{1-0}{2-0} = \frac{x(t)-0}{t-0}$$

$$\frac{1}{2} = \frac{x(t)}{t}$$

$$2x(t) = t$$

$$x(t) = \frac{t}{2} \; ; \quad 0 \le t \le 2$$

(2)



$$\frac{-1-1}{4-2} = \frac{x(t)-1}{t-2}$$

$$= \frac{-2}{2} = \frac{x(t)-1}{t-2}$$

$$(t-2)-2 = (x(t)-1)\,2$$

$$-2t+4 = 2x(t)-2$$

$$2x(t) = -2t+4+2$$

$$x(t) = \frac{-2t+6}{2}$$

$$x(t) = -t+3$$

$$2 \leq t \leq 4$$

$$\therefore \; x(t) = \begin{cases} t/2, & 0 \leq t \leq 2 \\ -t+3, & 2 \leq t \leq 4 \\ -1, & 4 \leq t \leq 6 \end{cases}$$

This is not a periodic signals

$\therefore$ is a energy signal

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_0^2 \left(\frac{t^2}{4}\right) dt + \int_2^4 (-t+3)^2 dt$$

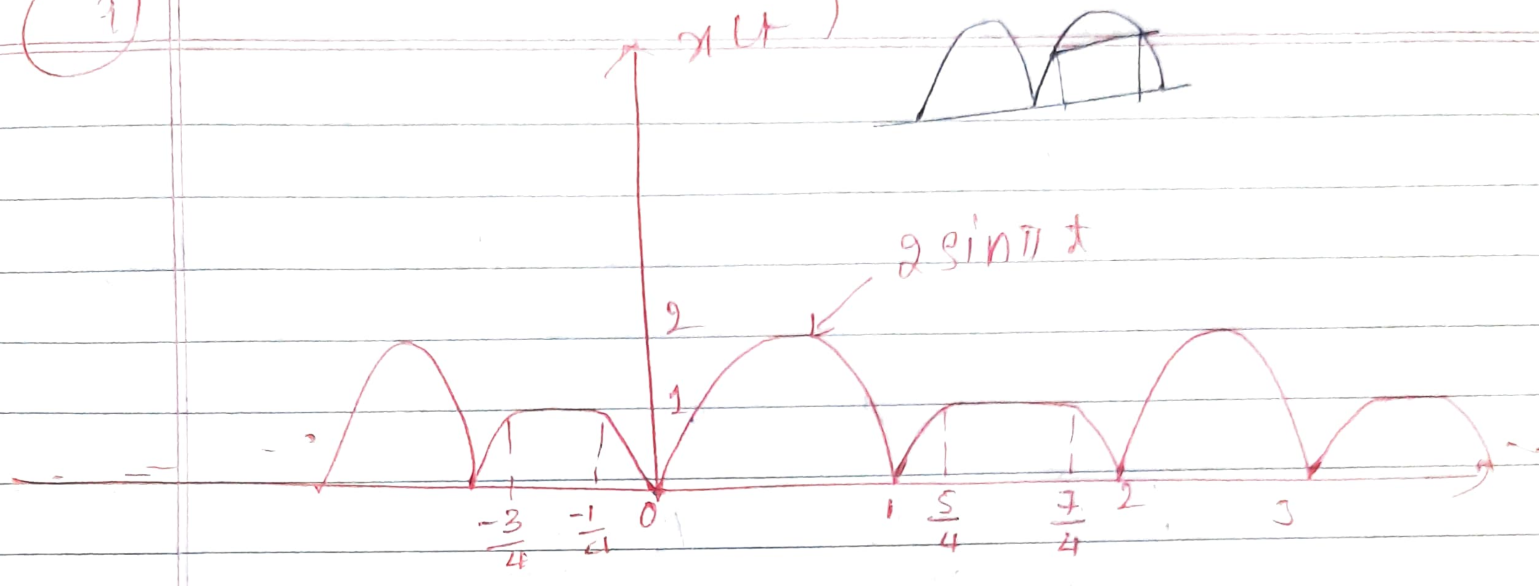
$$+ \int_4^6 (-1)^2 dt$$

$$= \frac{1}{4}\left(\frac{t^3}{3}\right)_0^2 + \left(\frac{t^3}{3} - \frac{6t^2}{2} + 9t\right)_2^4$$

$$+ (t)_4^6$$

$$= \frac{1}{12}(8) + \left[\frac{1}{3}(64-8) - 3(16-4)\right)$$

7



$$x(t) = \begin{cases} 2 \sin \pi t \,; & 0 \le t \le 1 \\ 2 \sin \pi (t-1); & 1 \le t \le 5/4 \\ & \frac{7}{4} < t < 2 \\ 1 \;; & \frac{5}{4} \le t \le \frac{7}{4} \end{cases}$$

x(t) is periodic with a fundamental period 2. ∴ it is a power signal

$$P = \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

$$= \frac{1}{2} \Bigg[\int_0^1 (2 \sin \pi t)^2 dt + \int_1^{5/4} [2 \sin \pi (t-1)]^2 dt + \int_{5/4}^{7/4} 1^2 dt + \int_{7/4}^{2} [2 \sin \pi (t-1)]^2 dt$$

$$\sin \pi t = \frac{1-\cos 2\pi t}{2}$$

$$= \frac{1}{2}\left[4\int_{0}^{1}\frac{1-\cos 2\pi t}{2}\,dt + 4\int_{1}^{5/4}\frac{1-\cos 2\pi(t-1)}{2}\,dt\right.$$

$$\left. + \int_{5/4}^{7/4} 1\,dt + 4\int_{7/4}^{2}\frac{1-\cos 2\pi(t-1)}{2}\,dt\right]$$

$$= \frac{1}{2}\left[2\left(t-\frac{\sin 2\pi t}{2\pi}\right)_{0}^{1} + 2\left(t-\frac{\sin 2\pi(t-1)}{2\pi}\right)_{1}^{5/4}\right.$$

$$\left. + \frac{1}{2}[t]_{5/4}^{7/4} + \left[t-\frac{\sin 2\pi(t-1)}{2\pi}\right]_{7/4}^{2}\right]$$

$$\left(1-\frac{\sin 2\pi}{2\pi}\right) - \left(0-\frac{\sin 0}{2\pi}\right) + \left(\frac{5}{4}-\frac{\sin 2\pi\times 5/4}{2\pi}\right)$$

$$-\left(1-\frac{\sin 2\pi}{2\pi}\right) + \frac{1}{2}\left(\frac{7}{4}-\frac{5}{4}\right) + \left(2-\frac{\sin 4\pi}{2\pi}\right)$$

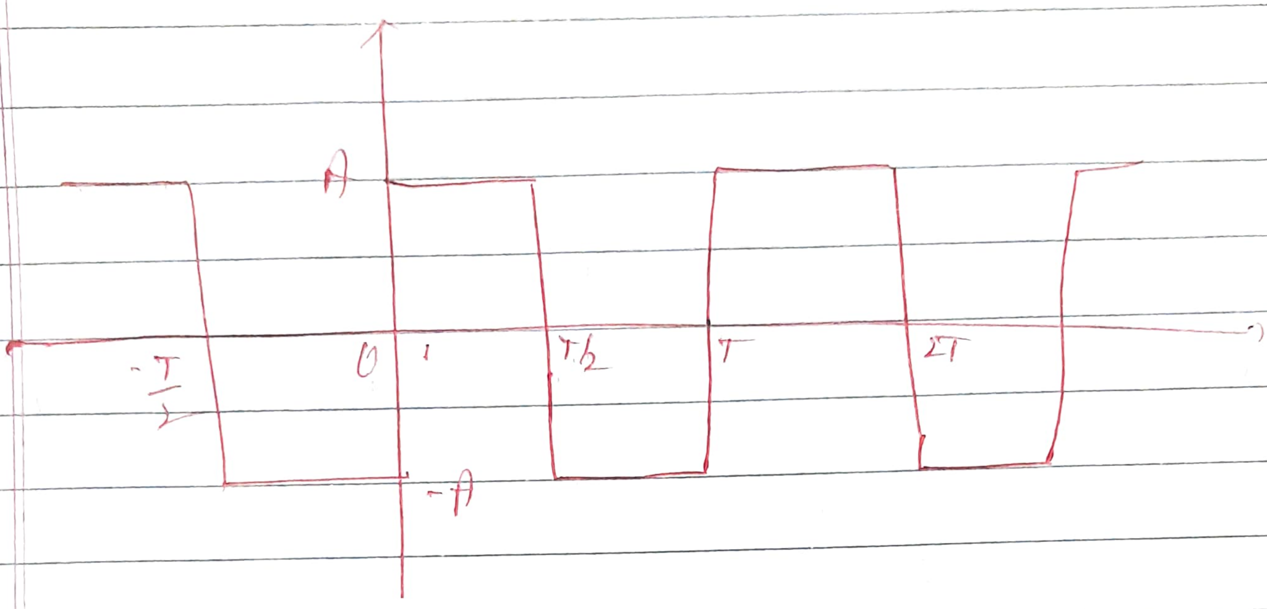
$$= 1 - 0 + 5/4 - 0.1591 - 1 + 0.25 + 2 - 7/4 - 0.1591$$

$$\boxed{= 1.4318}$$

$$\boxed{1.75}$$

⑯ Find the avg powe of the signal

Ⓐ

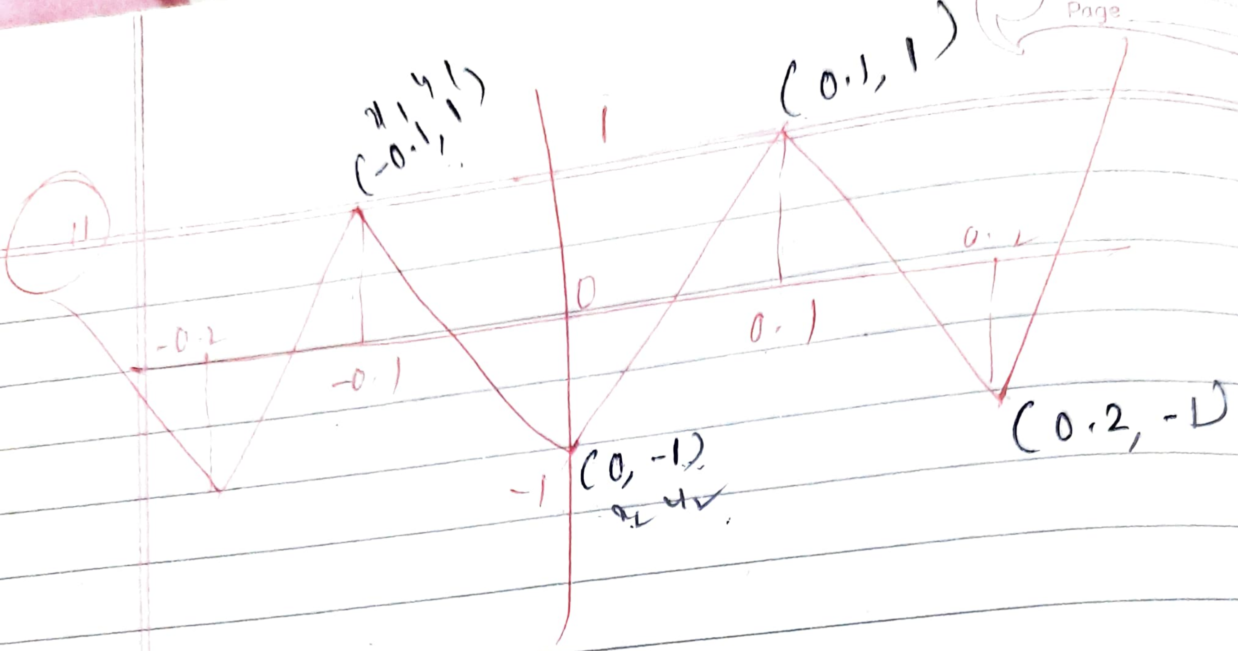


signal is periodic with fundamental period = T sec

$$x(t) = \begin{cases} A, & 0 < t < T/2 \\ -A, & T/2 < t < T \end{cases}$$

$$P_{avg} = \frac{1}{T}\int_0^T |x(t)|^2 dt$$

$$= \frac{1}{T}\int_0^T A^2 dt = \frac{A^2}{T}[t]_0^T = \underline{\underline{A^2}}$$



$$\frac{y_2-y_1}{x_2-x_1} = \frac{y-y_1}{x-x_1}.$$

$x_1 = 0,\ y_1 = -1,\ x_2 = 0.1,\ y_2 = 1$

$$x(t) = \begin{cases} 20t - 1, & 0 \le t \le 0.1 \\ -20t & ,-0.1 \le t \le 0 \end{cases}$$

period of $x(t) = 0.2$sec

$P_{avg} = \underline{\underline{0}}$

⑩ $x(n) = e^{j\left(\frac{\pi}{2}n + \frac{\pi}{6}\right)}$

~~Att~~ $|x(n)| = 1$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} 1 = \infty$$

$$P = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} 1$$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \times 2N+1$$

$$= \underline{\underline{1}}$$

hence a power signal

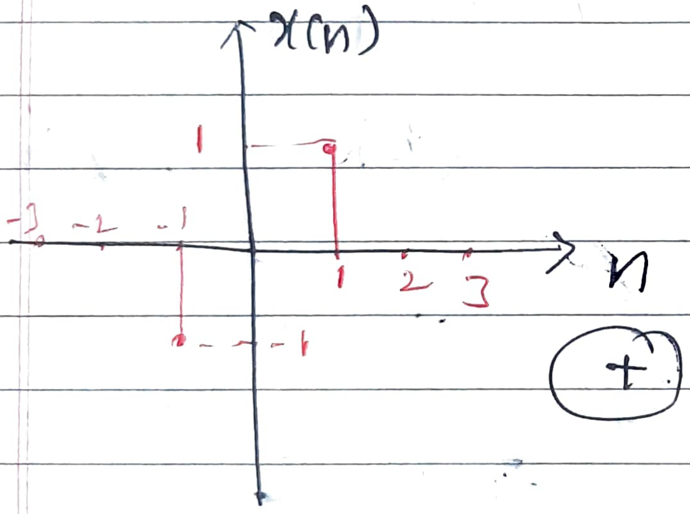
1) Problems on operation of signals

① A discrete-time signal is defined by

$$x(n) = \begin{cases} 1, & n = 1 \\ -1, & n = -1 \\ 0, & n = 0 \ \& \ |n| > 1 \end{cases}$$

Ⓣ Find y(n) which is defined by

$y(n) = x(n) + x(-n)$



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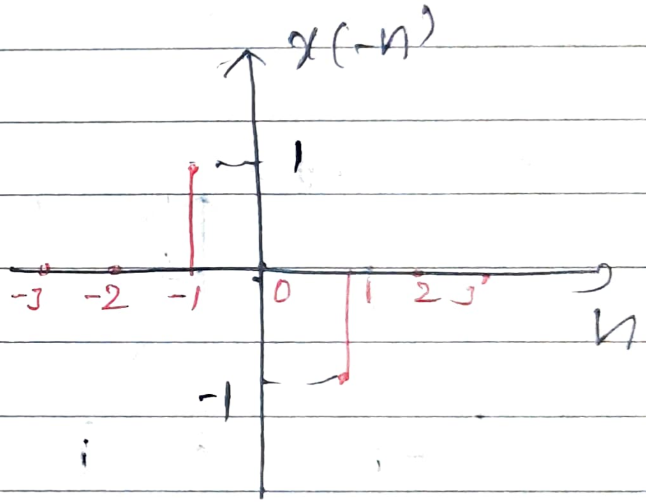
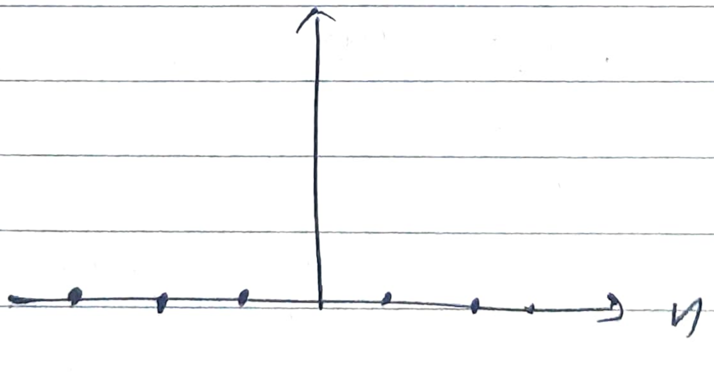
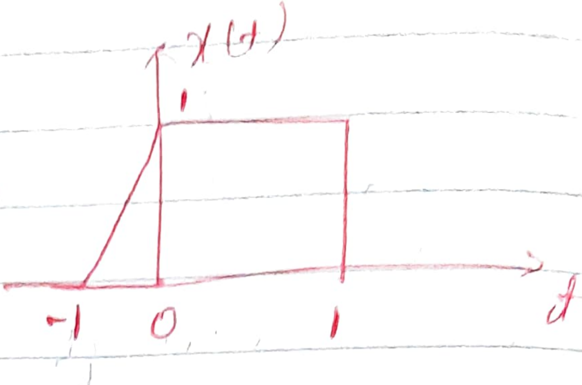




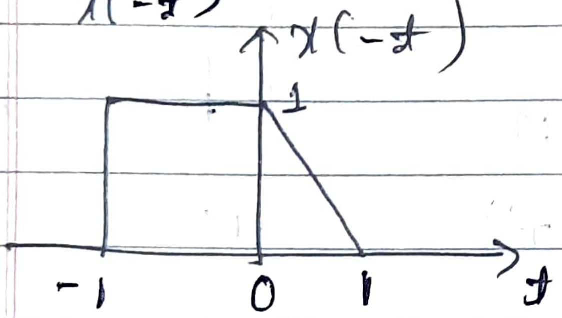
$y(n) = 0 \ \forall \ n$

② For signal x(t) shown below sketch the foll signals

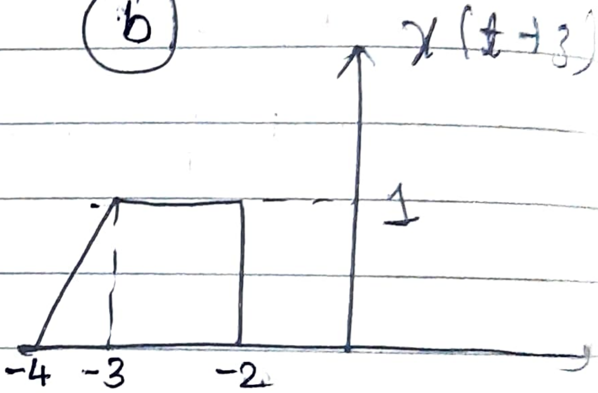
ⓐ $x(-t)$
ⓑ $x(t+3)$
ⓒ $x(2t-2)$
ⓓ $x(0.5t-1)$
ⓔ $x(2-2t)$
ⓖ $-x(3t+4)$ ⓕ $x(-t+3)$.



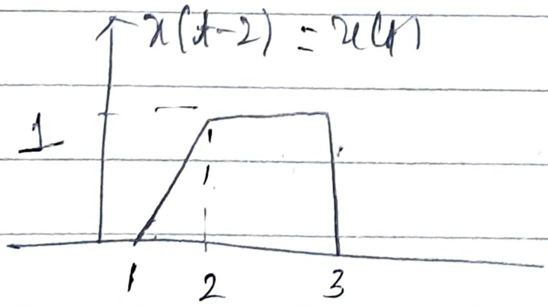
ⓐ $x(-t)$



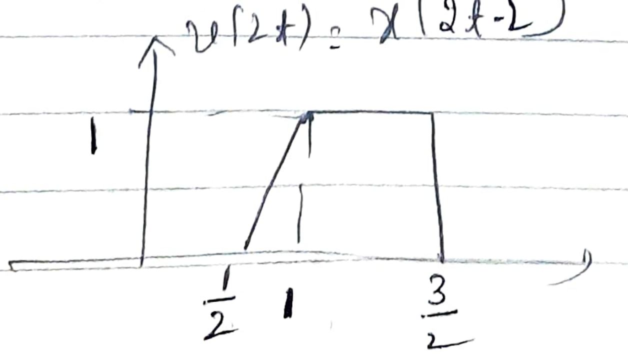
ⓑ



ⓒ $x(2t-2)$ $\quad x(t) \xrightarrow{shift} x(t-2) \xrightarrow{scaling} x(2t-2)$

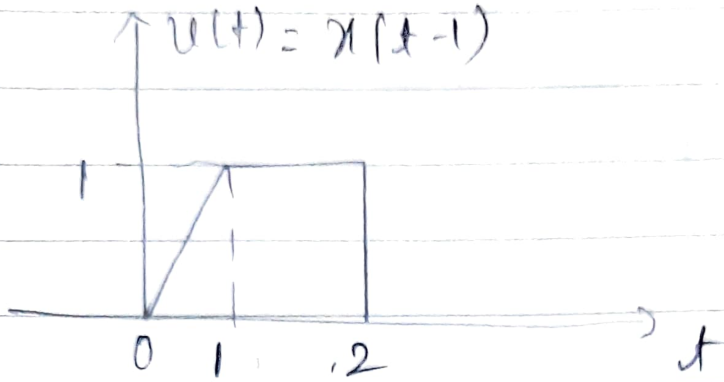


Then perform scaling on v(t)

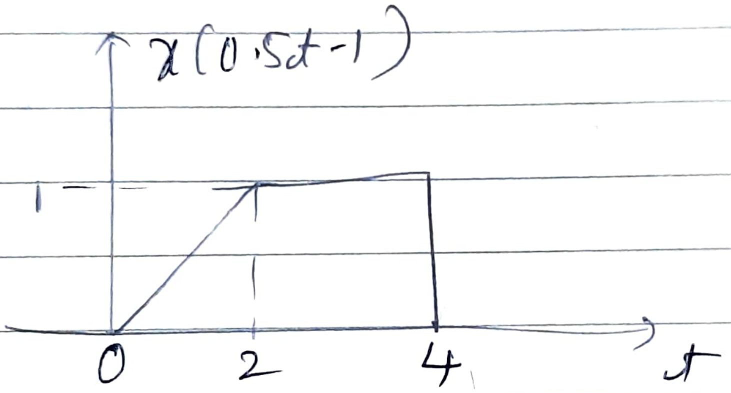


ⓓ $x(0.5t-1)$

(i) first perform $x(t-1) = v(t)$



Then perform $v(0.5t) = x(0.5t-1)$



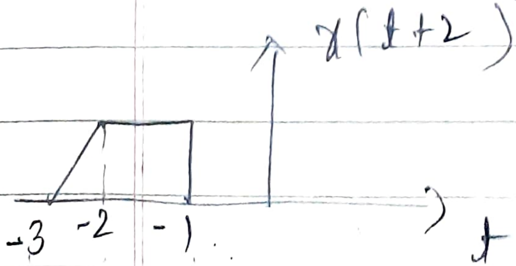
$x(-at+b)$

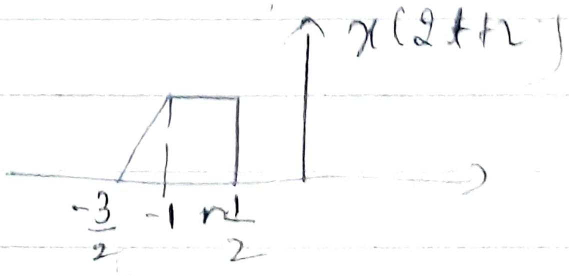
ⓔ $x(2-2t) = x(-2t+2)$ $x(-2t-(-2))$

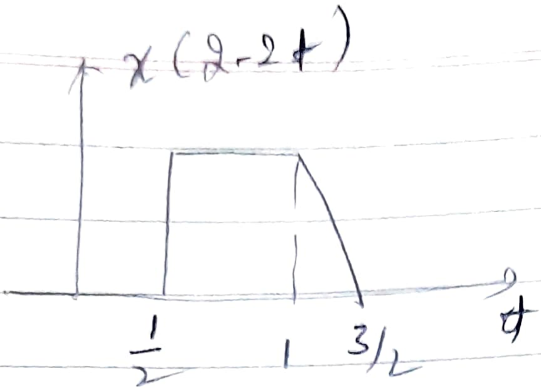
(i) first perform $x(t+2) = v(t)$

(ii) Then perform $x(2t+2) = v(2t)$

(iii) Then perform $x(-2t+2) = v(-2t)$

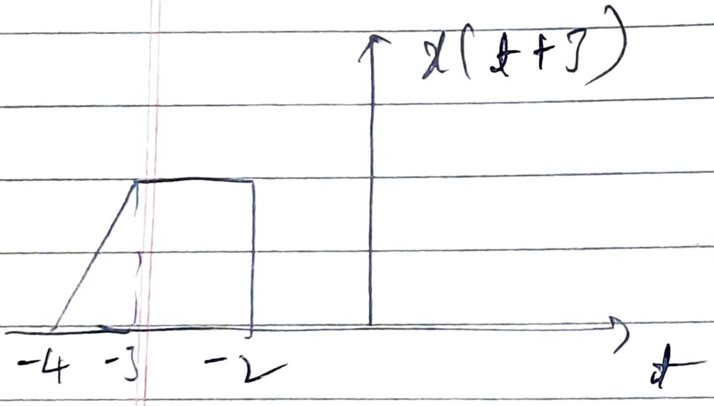


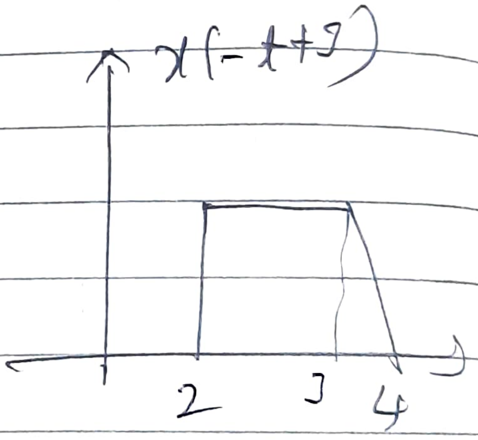




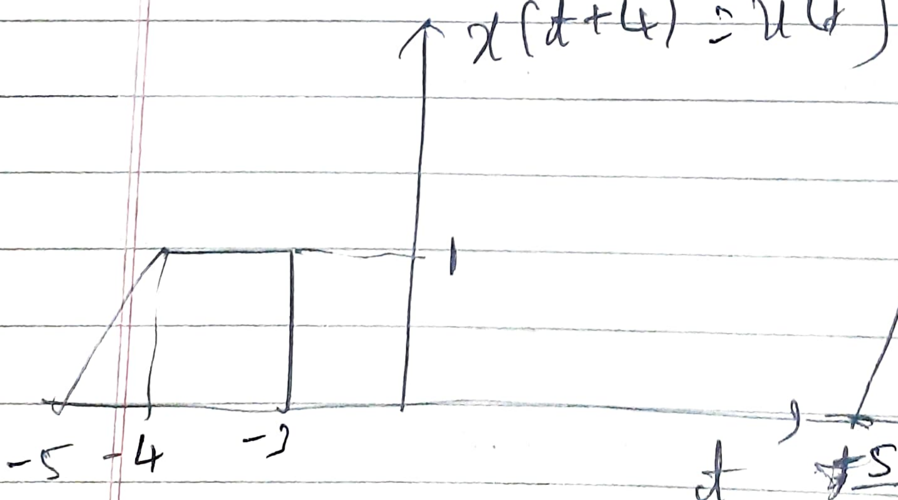
(f) $x(-t+3)$

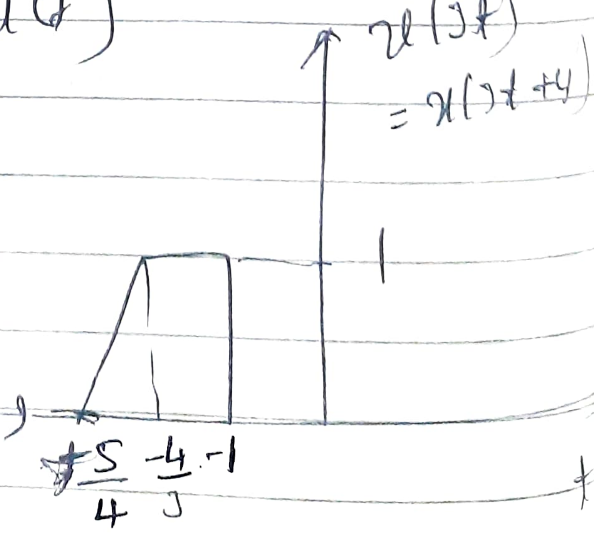
(i) first perform $x(t+3)$ then $x(-t)$





(g) $x(3t+4)$





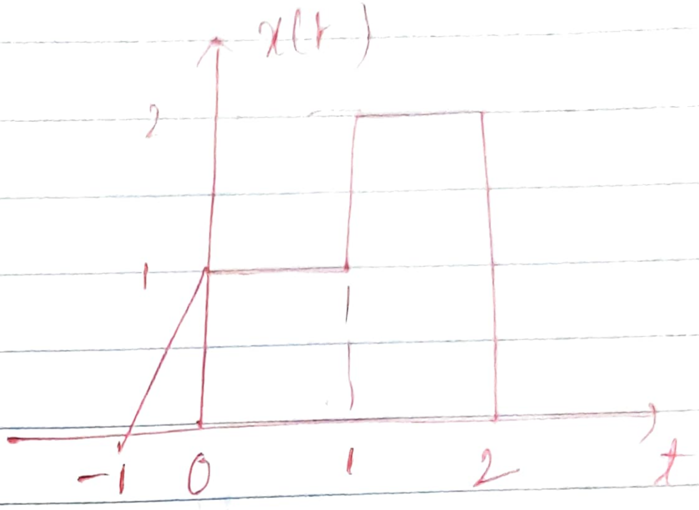
② Given $x(t)$ sketch the foll

ⓐ $x(-t)$

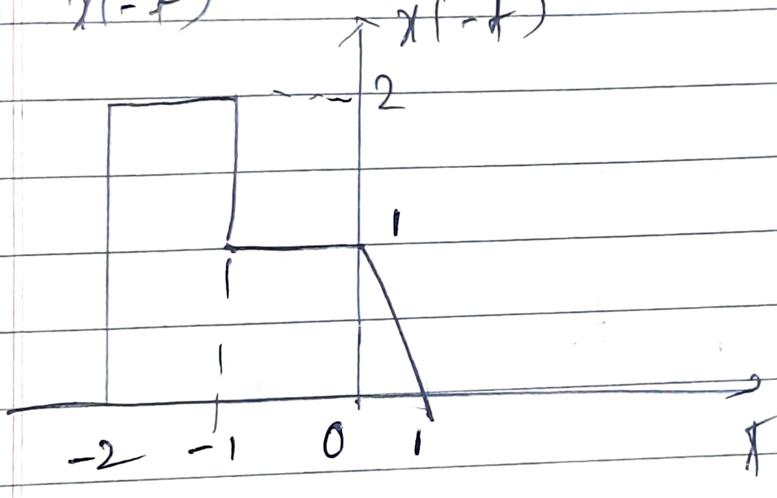
ⓑ $x(t - 3/2)$

ⓒ $x(2t - 1)$

ⓓ $x(4 - t)$



ⓐ $x(-t)$



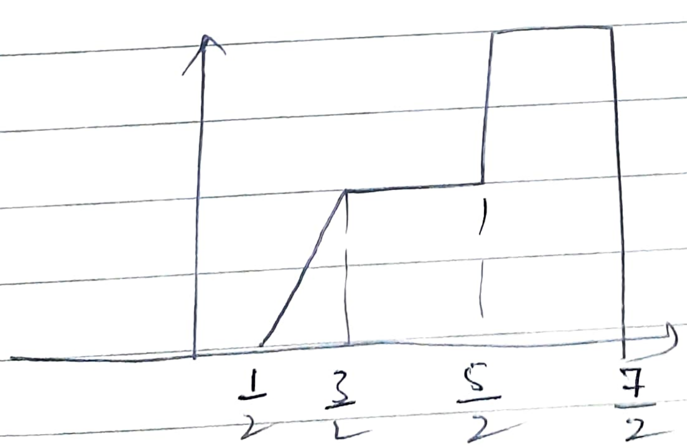
ⓑ $x(t - 3/2)$

$-1 \to -1 + \frac{3}{2} = \frac{-2+3}{2} = \frac{1}{2}$

$1 \to 1 + \frac{3}{2} = \frac{2+3}{2} = \frac{5}{2}$

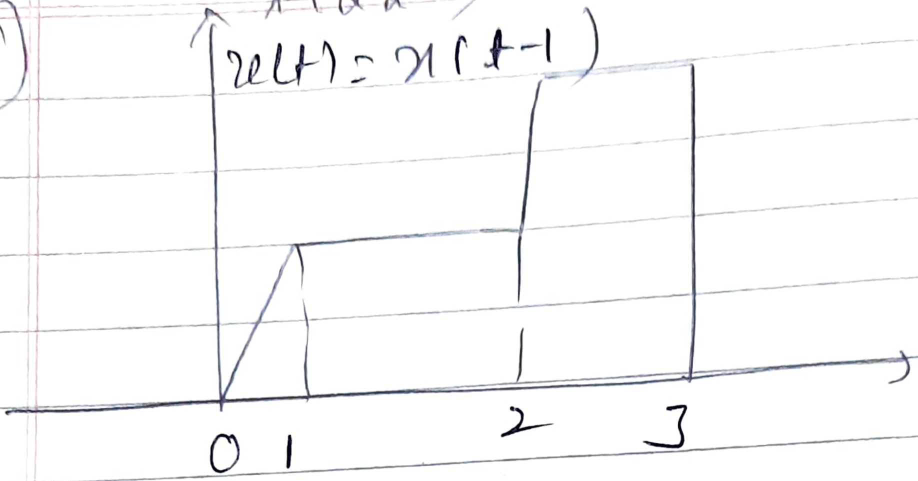
$2 \to 2 + \frac{3}{2}$

$= \frac{4+3}{2} = \frac{7}{2}$

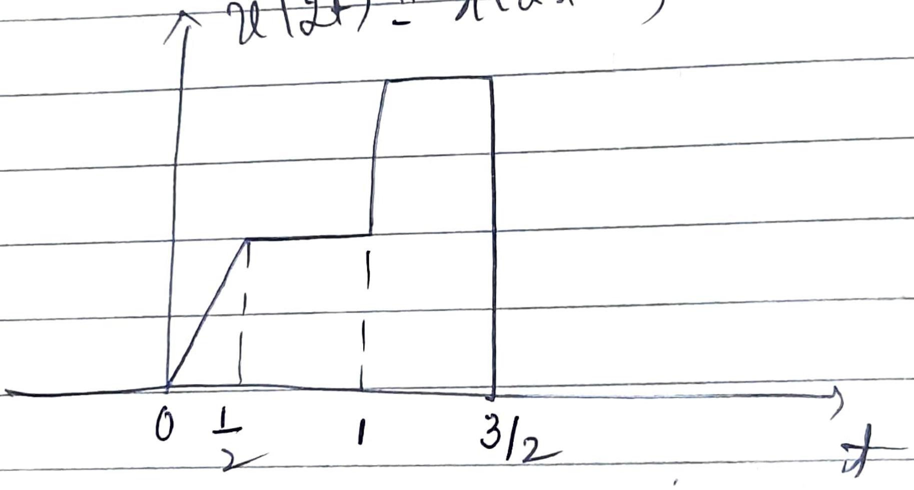


ⓒ $x(2t-1)$

$x(t) = x(t-1)$

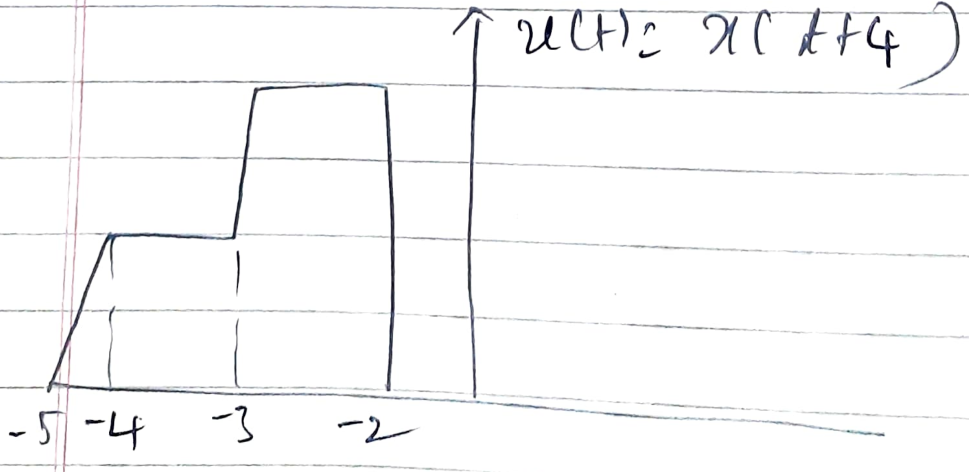


$x(2t) = x(2t-1)$

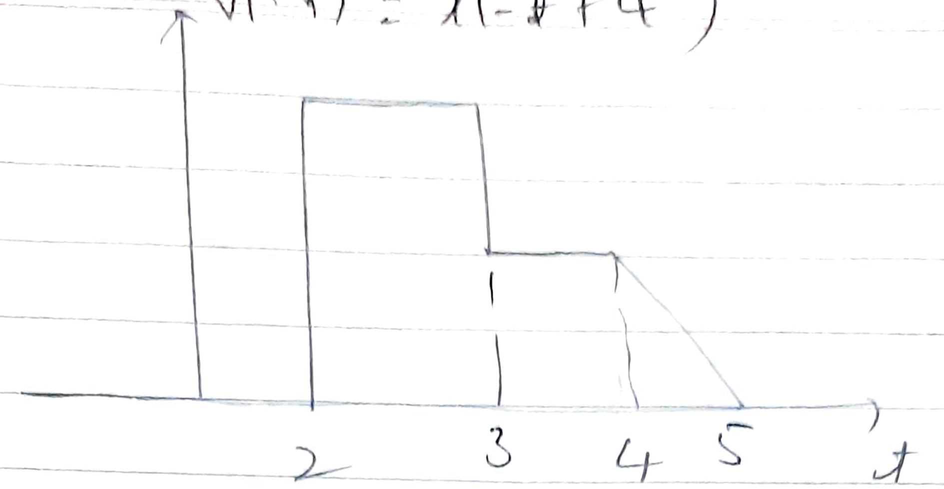


ⓓ $x(4-t) = x(-t+4)$

$x(t) = x(t+4)$



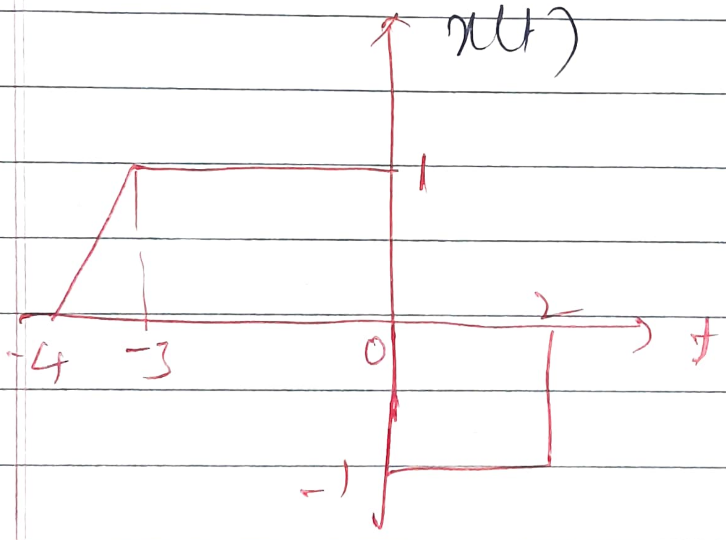
$v(-t) = x(-t+4)$



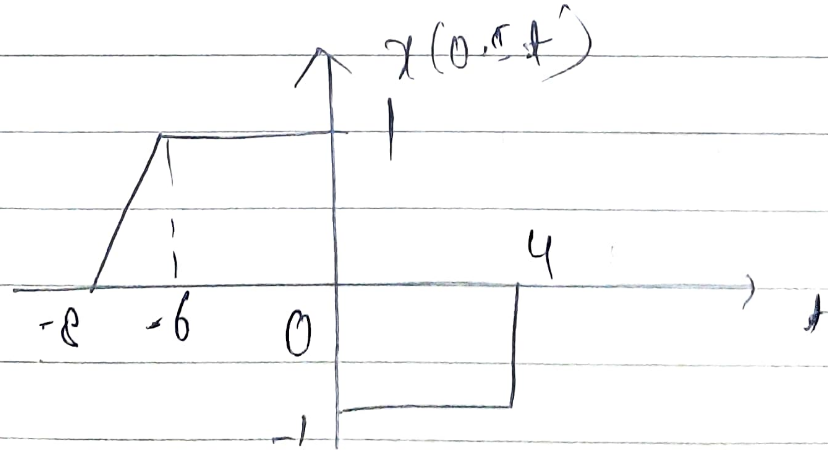
⑤ The signal $x(t)$ is shown below sketch

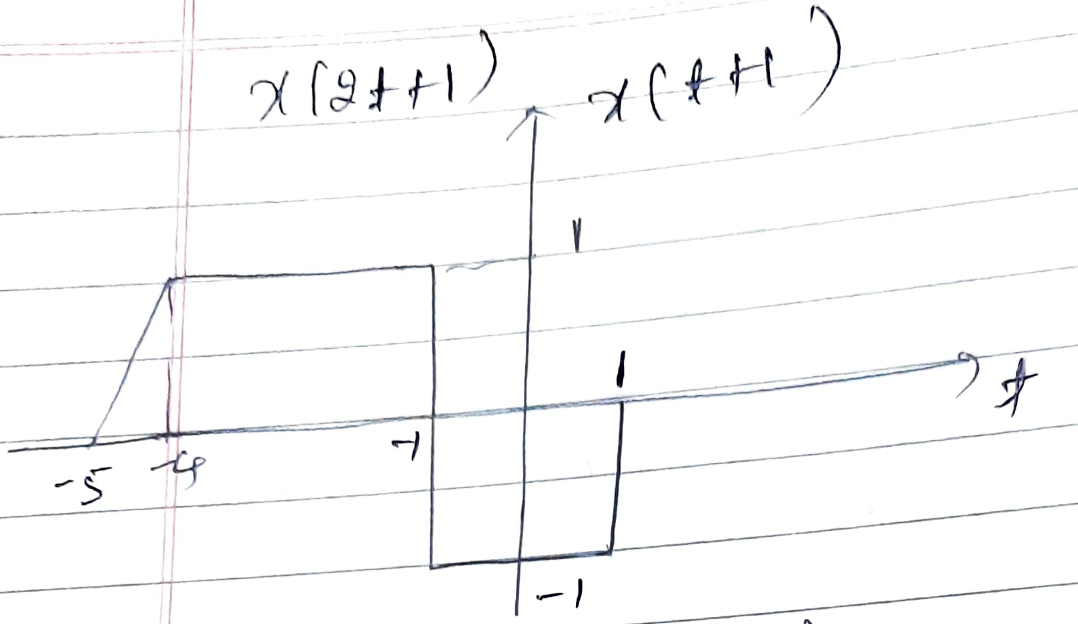
ⓐ $x(0.5t)$ ⓑ $x(2t+1)$ ⓒ $x(3-t)$

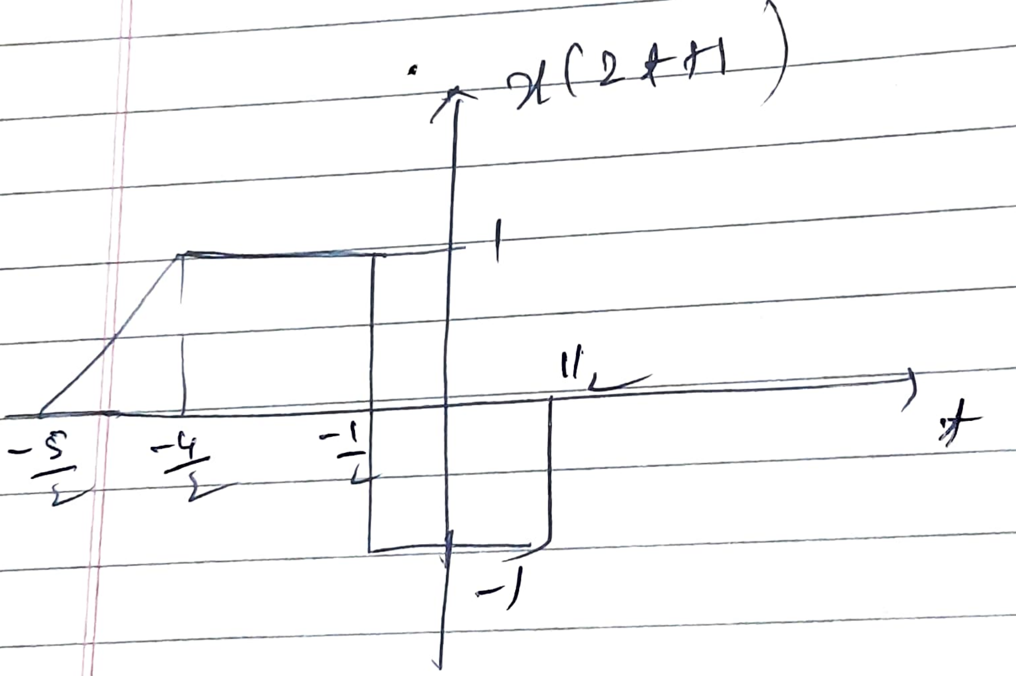
ⓓ $x(t+5)$



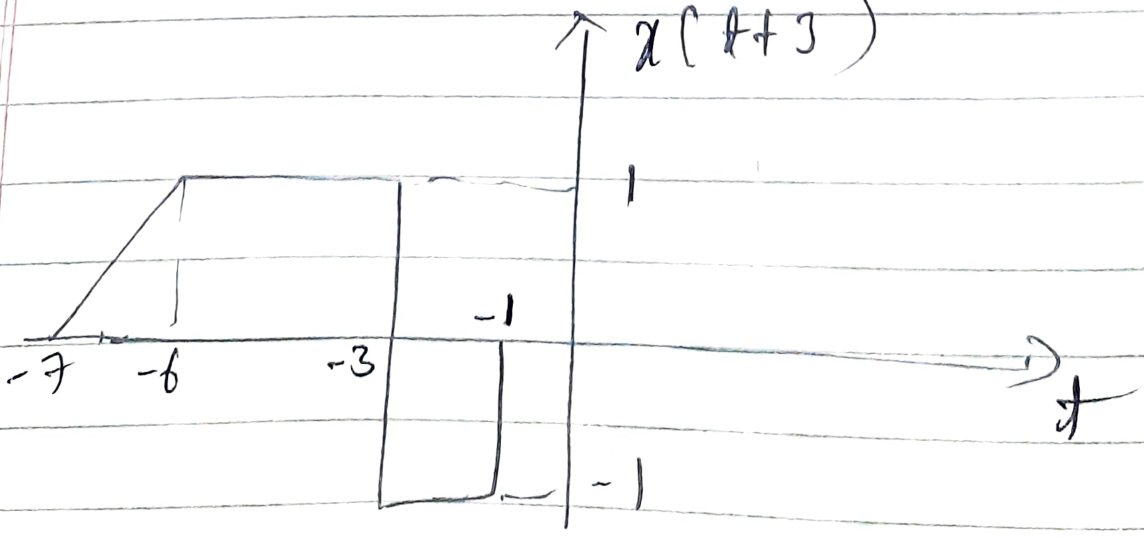
ⓐ

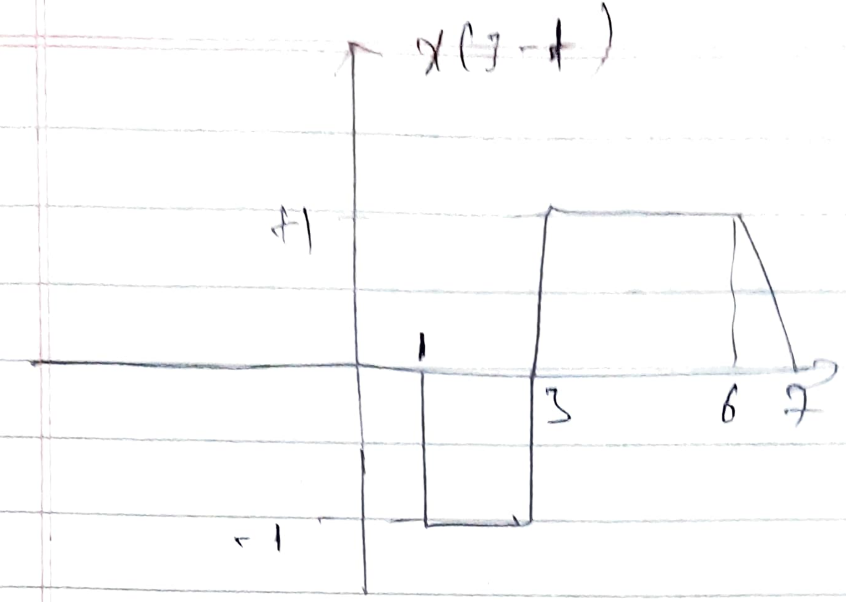




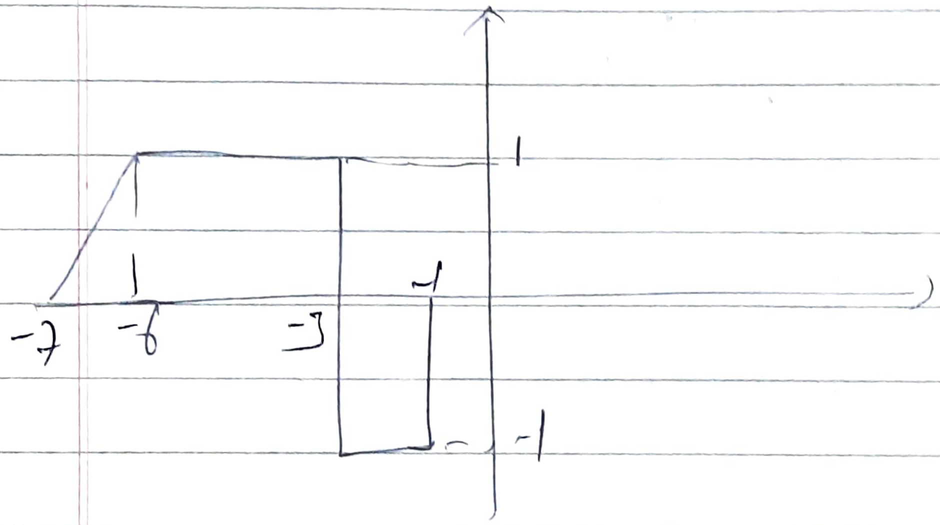


ⓒ $x(3-t) = x(-t+3)$





ⓓ x(t+3)



④ Given $x(n) = \{1, 4, 7, 9, 8\}$

$y(n) = \{3, 5, \underset{\uparrow}{0}, 6, 2\}$

Sketch the foll'

ⓐ $3x(n) + 2y(n)$

ⓑ $3x(n) - 2y(n)$

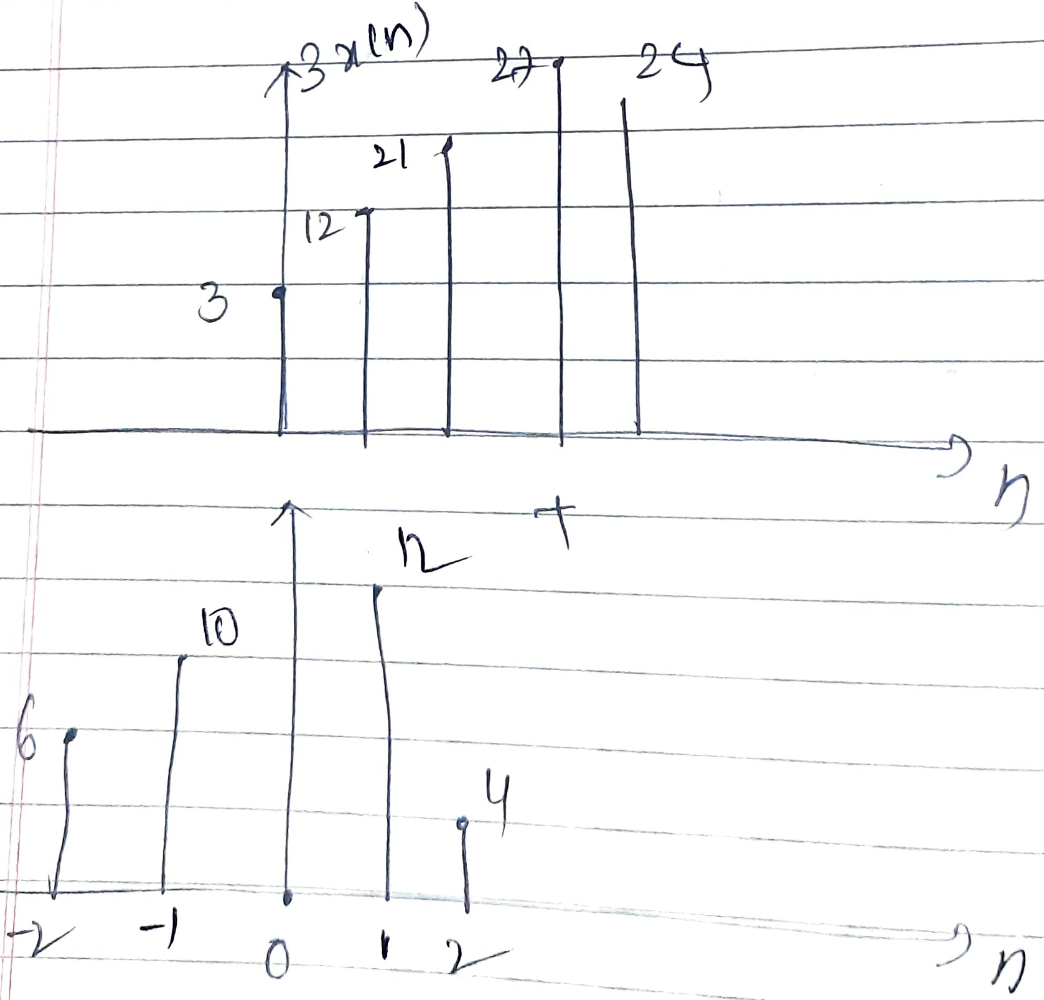
ⓒ $x(n) \cdot y(n-1)$

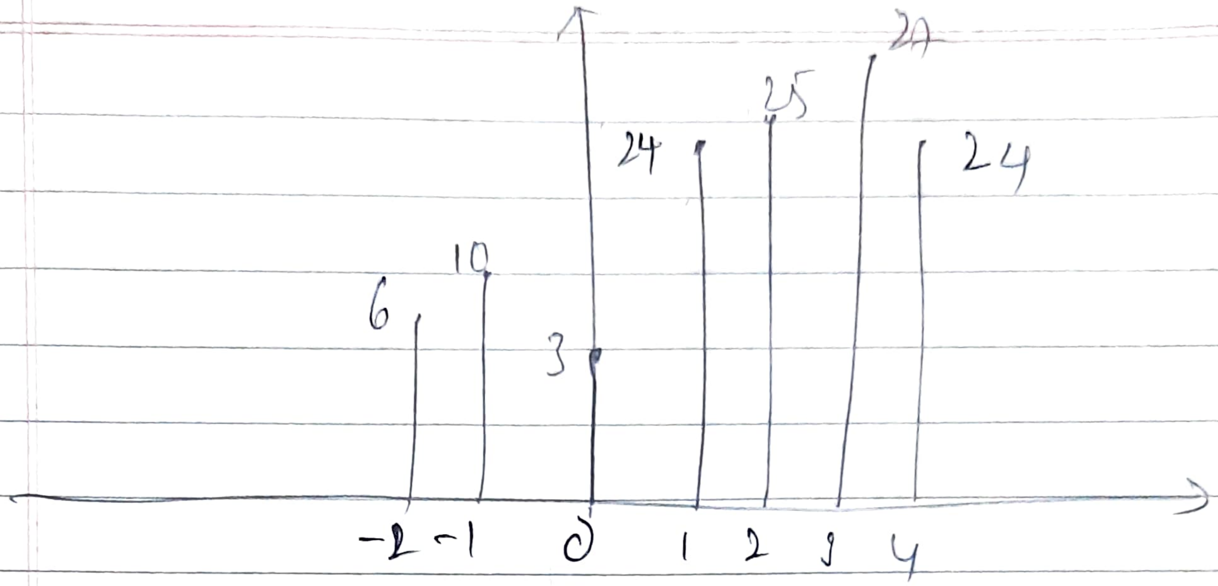
ⓐ $3x(n) + 2y(n)$

$3x(n) = \{3, 12, 21, 27, 24\}$

$2y(n) = \{6, 10, \underset{\uparrow}{0}, 12, 4\}$

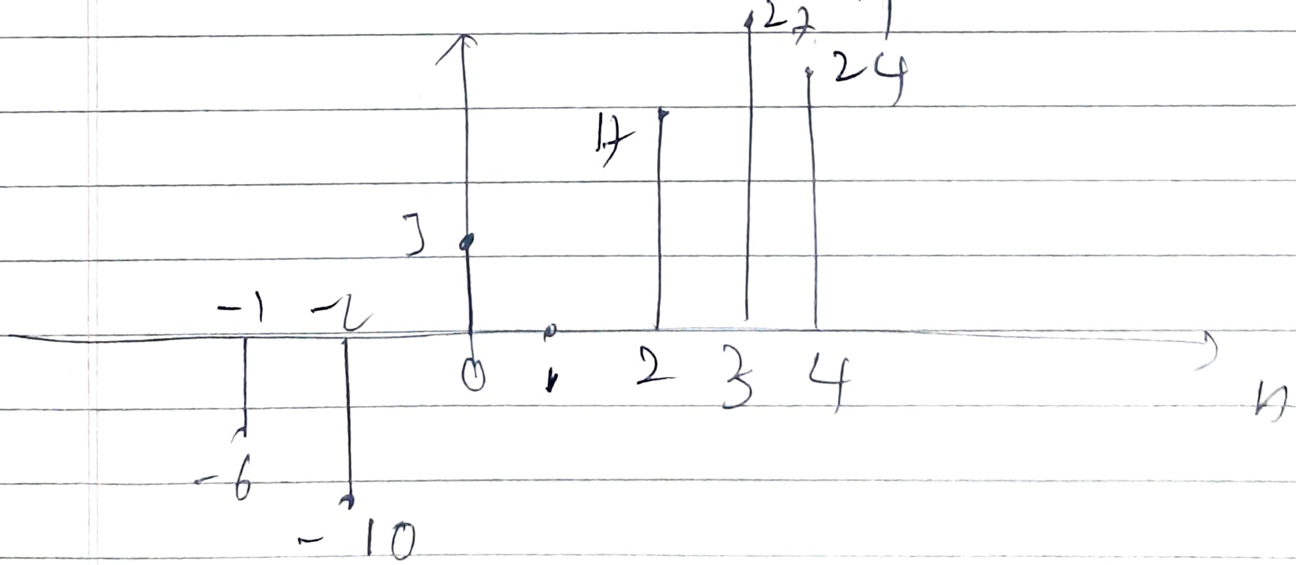
$3x(n) + 2y(n)$





$\{6, 10, \underset{\uparrow}{3}, 24, 25, 27, 24\}$

ⓑ $3x(n) - 2y(n)$
$= \{-6, -10, \underset{\uparrow}{3}, 0, 17, 27, 24\}$

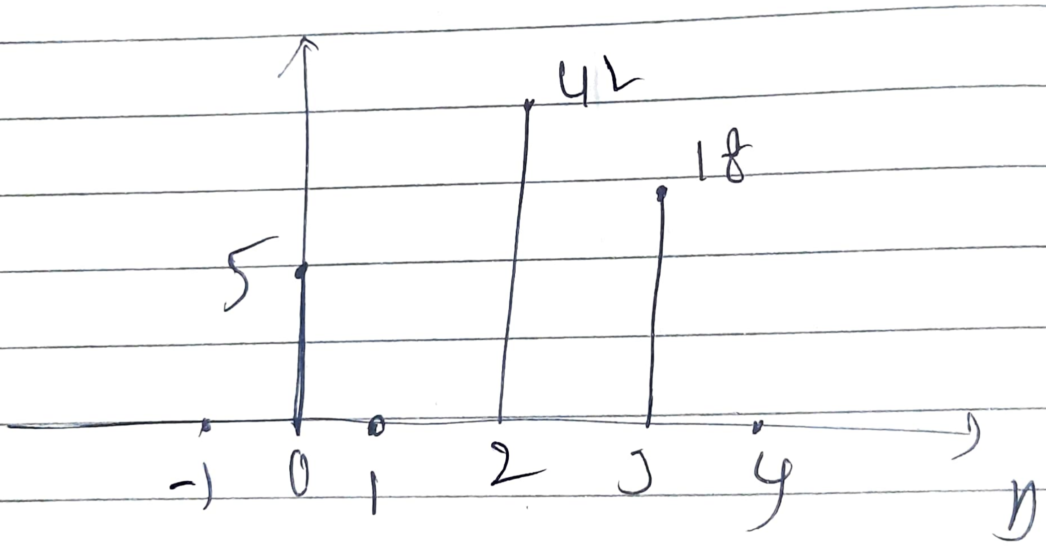


$x(n) \cdot y(n-1)$

$x(n) = \{1, 4, 7, 9, 8\}$

$y(n-1) : \{3, \underset{\uparrow}{5}, 0, 6, 2\}$

$(n-1) : \{0, \underset{\uparrow}{5}, 0, 42, 18, 0\}$



Given

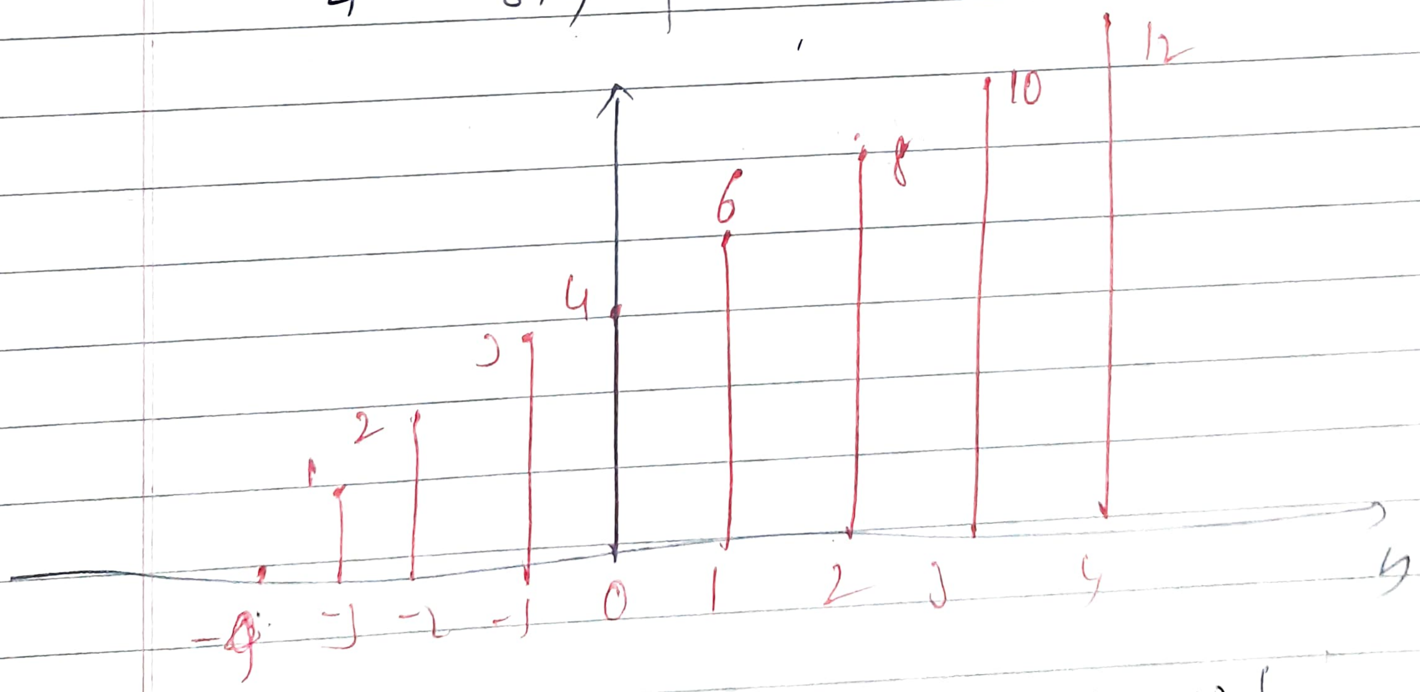
$$x(n) = \begin{cases} n+4, & -4 \le n < -1 \\ 2n+4, & -1 \le n \le 4 \\ 0, & \text{elsewhere} \end{cases}$$

sketch

$y_1(n) = x(4-n)$

$y_2(n) = x(2n-3)$

n	n+4	x(n)
-4	-4+4	0
-3	-3+4	+1
-2	-2+4	2
-1	-1+4	3
	2n+4	
0	0+4	4
1	2+4	6
2	4+4	8
3	6+4	10
4	8+4	12



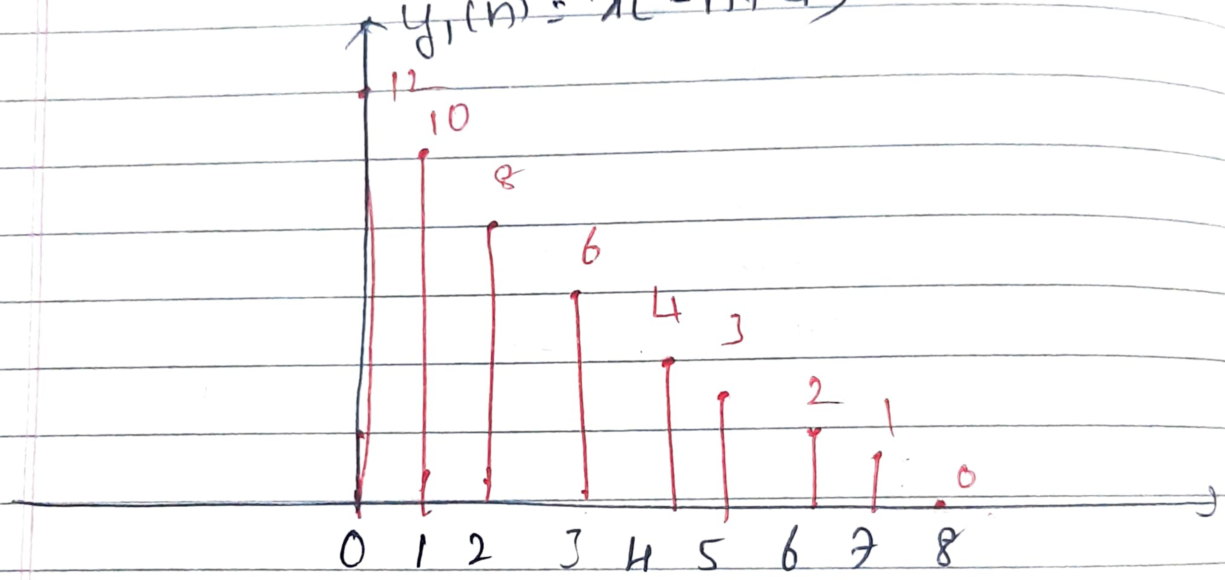
$x(n) = \{0, 1, 2, 3, \underset{\uparrow}{4}, 6, 8, 10, 12\}$

(a) $y_1(n) : x(4-n) = x(-n+4)$

$x(n) = x(n+4)$

=	0	1	2	3	4	6	8	10	12
	-8	-7	-6	-5	-4	-3	-2	-1	↑

$y_1(n) = x(-n+4)$

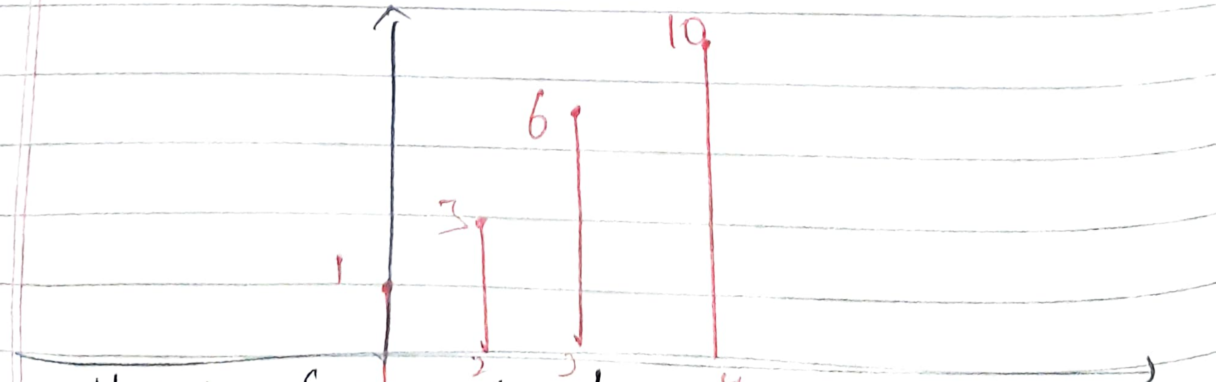


② $y_2(n) = x(2n-3)$

$x(n) = x(n-2)$

$y_2(n) = x(2n)$

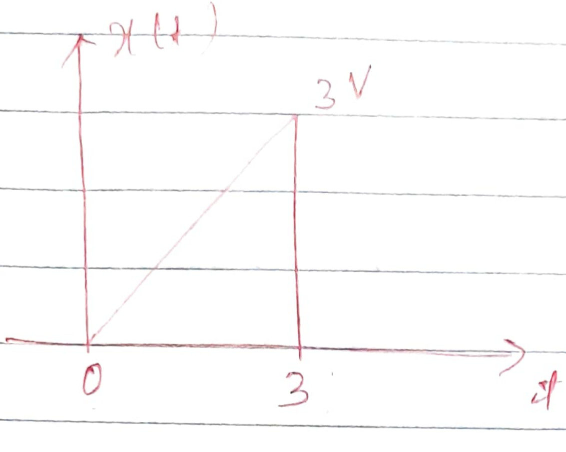
=	0	1	2	3	4	6	8	10
	-1	↑	1	2	3	4	5	6

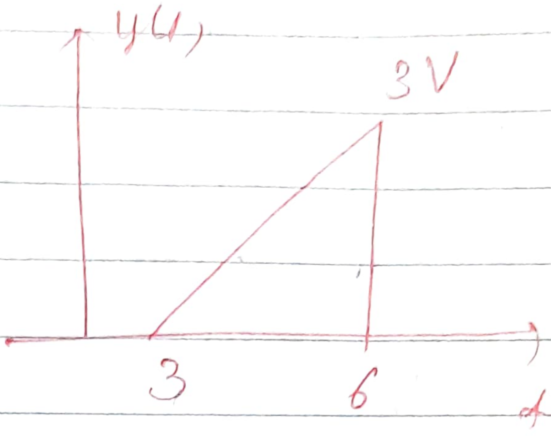


$y_2(n) = \{ \underset{\uparrow}{1}, 3, 6, 10 \}$

Ⓐ Express y(t) interms of x(t)

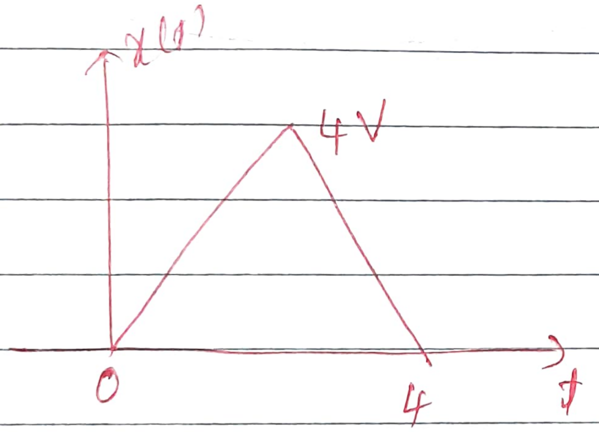
①

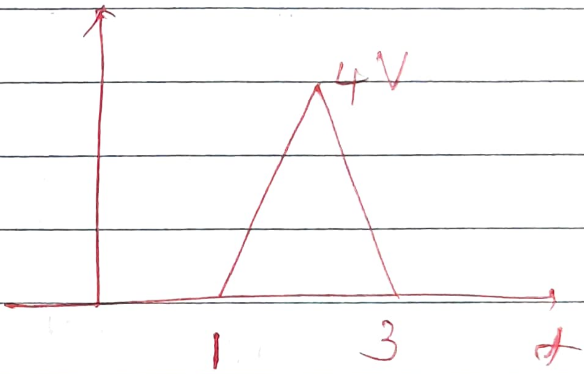


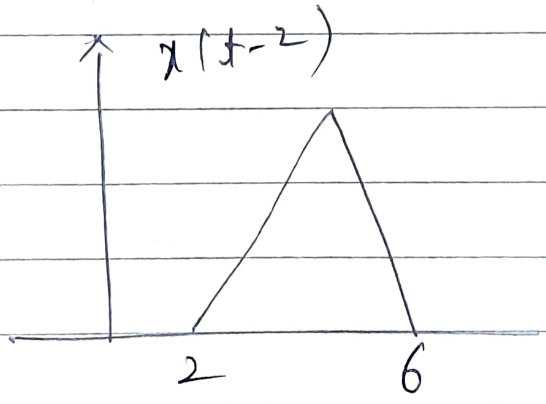


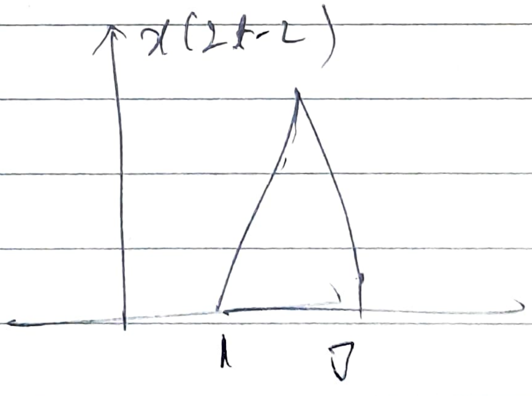
$$y(t) = x(t-3)$$

②



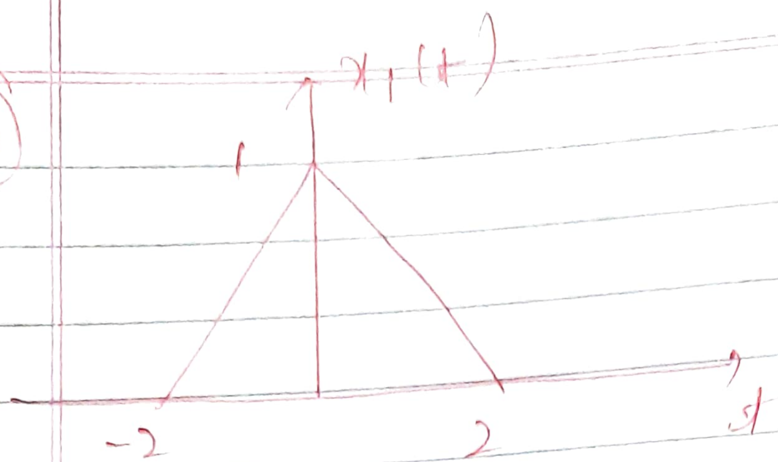


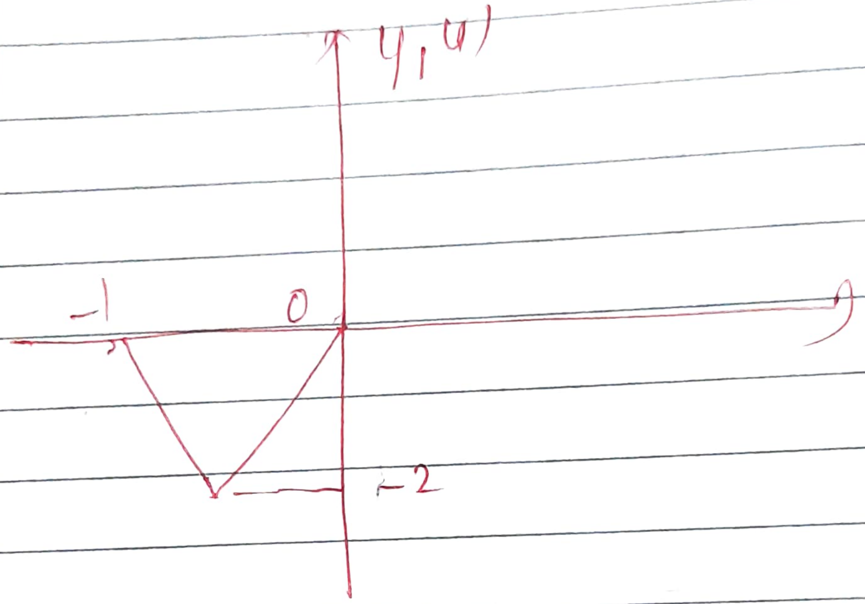


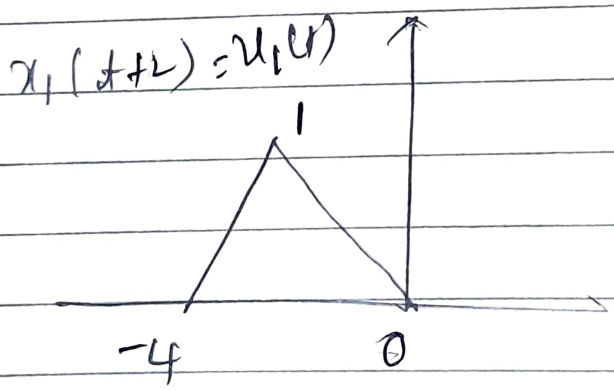


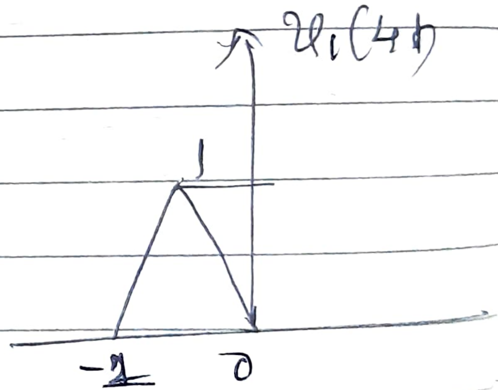
$$y(t) = x(2t-2)$$

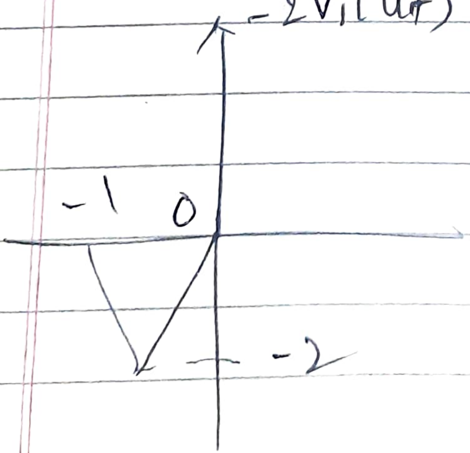
③





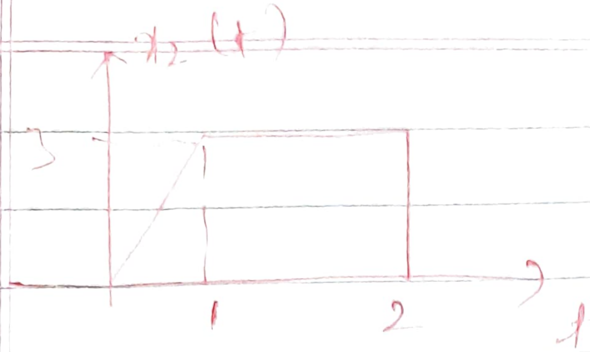


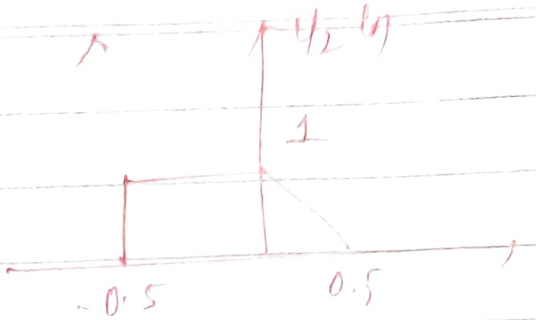


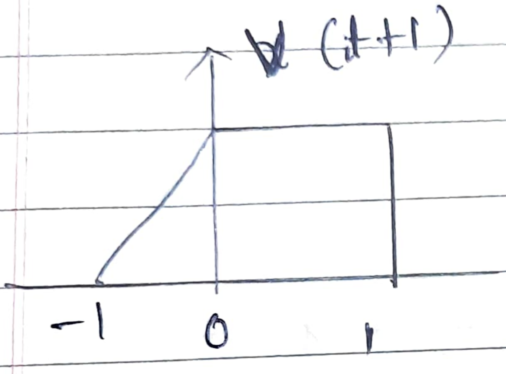


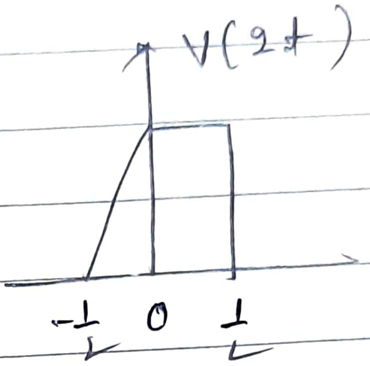
$$\boxed{y(t) = -2x_1(4t+2)}$$

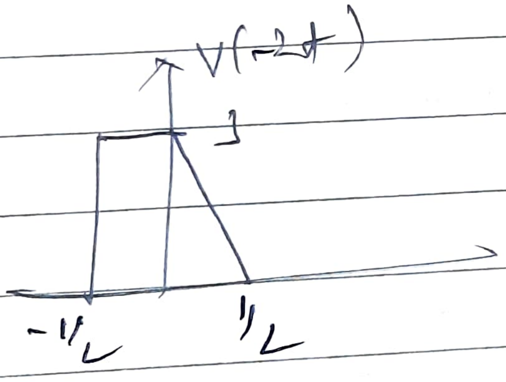
④

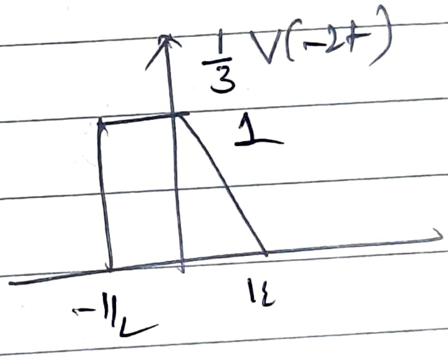












$$\boxed{y_2(t) = \frac{1}{3} x_2\left[-2t+1\right]}$$

II Problems on even & odd signals

① Find the even & odd parts of the foll signals

① $x(t) = e^{jt}$

$x(-t) = e^{-jt}$

WKT $x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right]$

& $x_0(t) = \frac{1}{2}\left[x(t) - x(-t)\right]$

$x_e(t) = \frac{1}{2}\left[e^{jt} + e^{-jt}\right]$

$= \underline{\underline{\cos t}}$

$x_0(t) = \frac{1}{2}\left[e^{jt} - e^{-jt}\right]$

$= \underline{\underline{j \sin t}}$

$\cos x = \frac{e^{jx} + e^{-j}}{2}$

$\sin x = \frac{e^{jx} - e^{-j}}{2j}$

① $x(t) = 4 + 3t + 5t^2 + 8t^3 + 9t^4$

$x(-t) = 4 - 3t + 5t^2 - 8t^3 + 9t^4$

$[x(t) + x(-t)] = 8 + 10t^2 + 18t^4$

$x_e(t) = \frac{x(t) + x(-t)}{2}$

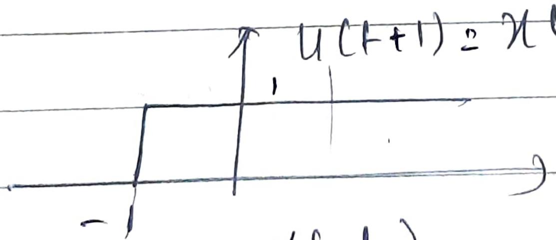
$= 4 + 5t^2 + 9t^4$

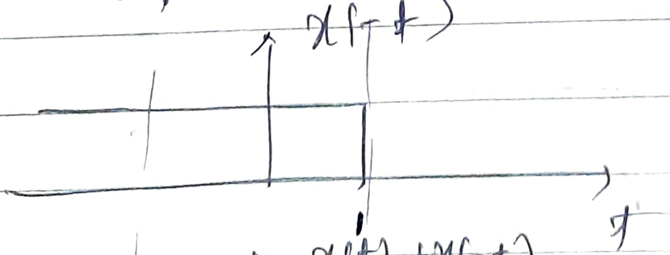
$x_0(t) = \frac{1}{2}[x(t) - x(-t)]$

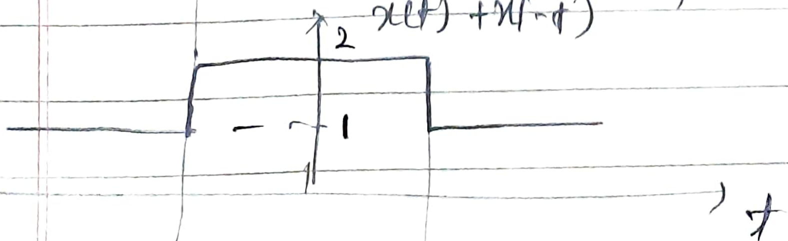
$= \underline{\underline{3t + 8t^3}}$

③ $x(t) = u(t+1)$

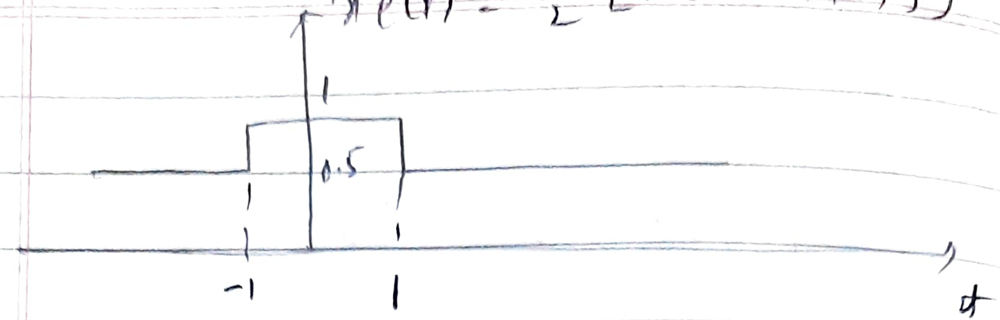
$x(-t) = u(-t+1)$

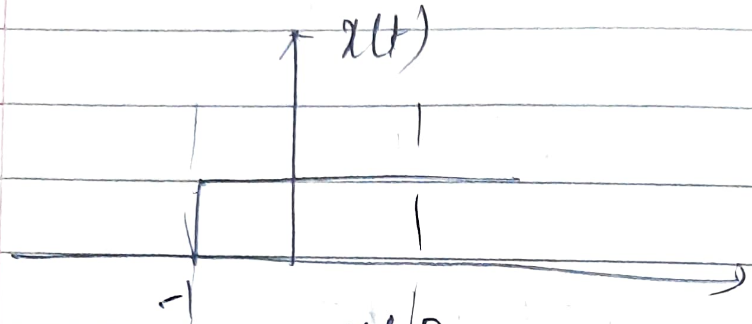


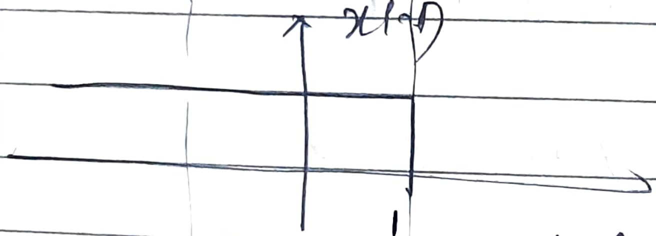


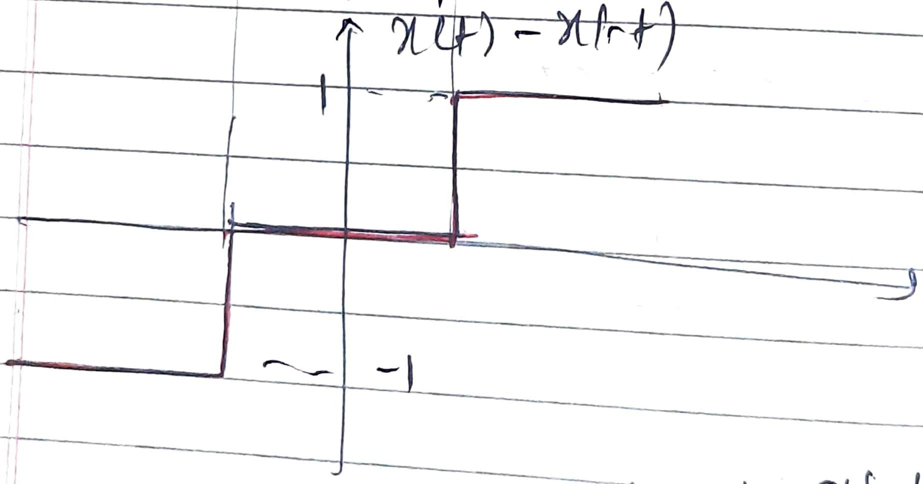


$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$

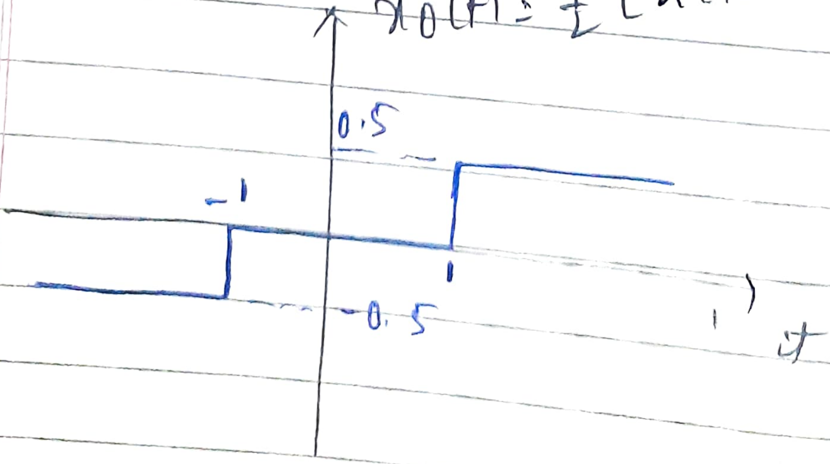








$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$



④ $x(n) = \{-1, \underset{\uparrow}{2}, -3, 4, -5\}$

$x(-n) = \{-5, 4, -3, \underset{\uparrow}{2}, -1\}$

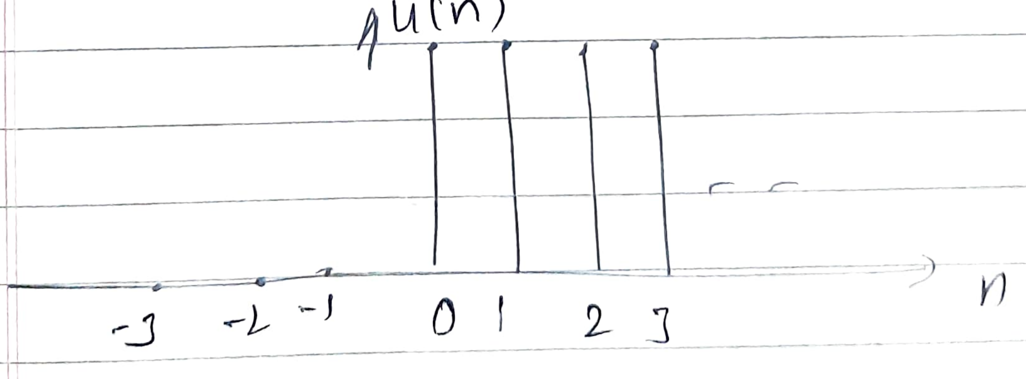
$x(n) + x(-n) = -5, 4, -4, 4, -4, 4, -5$

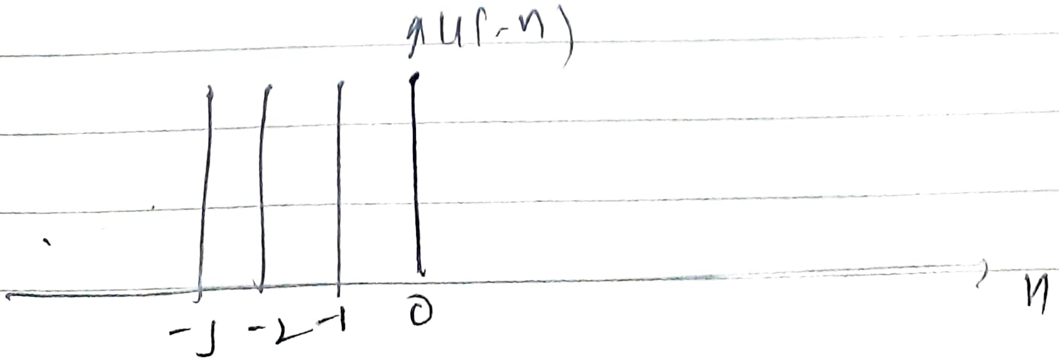
$x_e(n) = \{-\frac{5}{2}, 2, -2, 2, -2, 2, -\frac{5}{2}\}$

$x(n) - x(-n)$
$= \{5, -4, 2, \underset{\uparrow}{0}, -2, 4, -5\}$

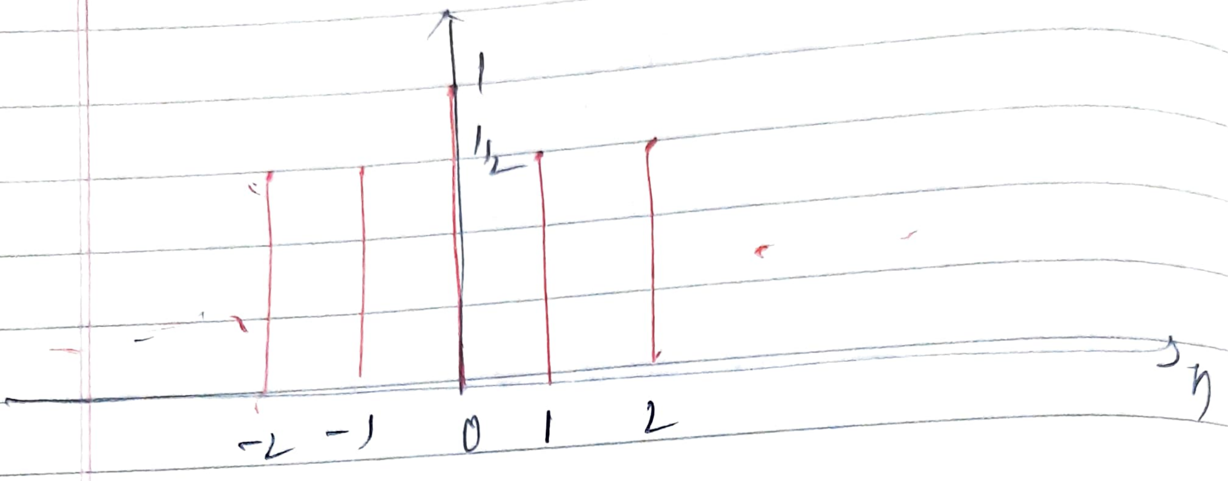
$x_0(n) = \{\frac{5}{2}, -2, 1, 0, -1, 2, -\frac{5}{2}\}$

⑤ $x(n) = u(n)$
$x(-n) = u(-n)$

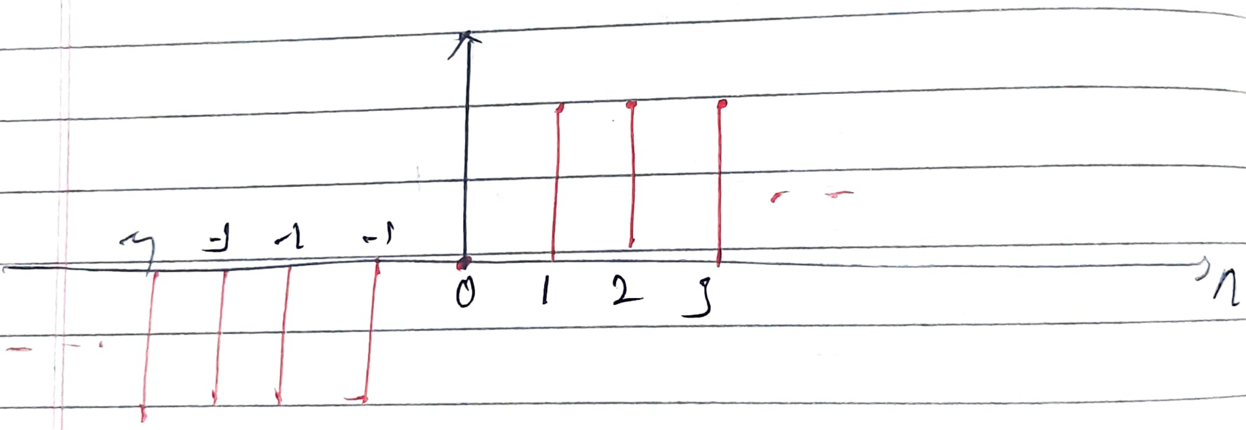




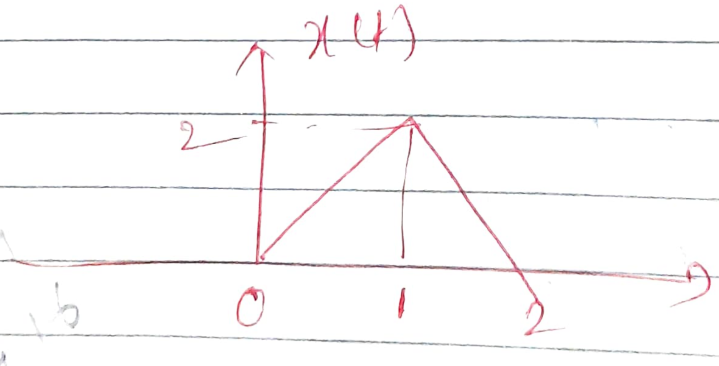
$$x_e(n) = \frac{1}{2}\left[x(n) + x(-n)\right]$$

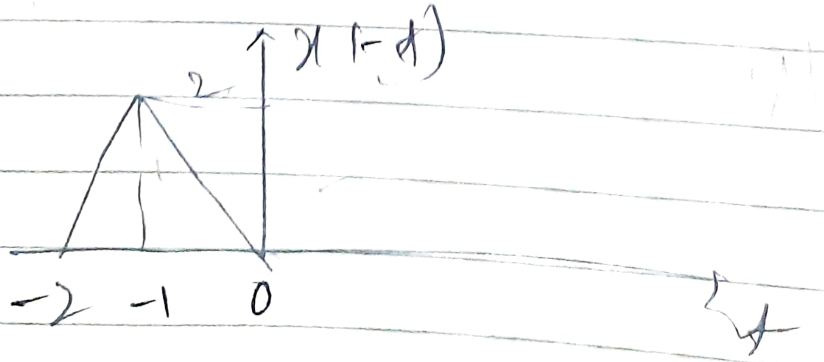


$$x_o(n) = \frac{1}{2}\left[x(n) - x(-n)\right]$$

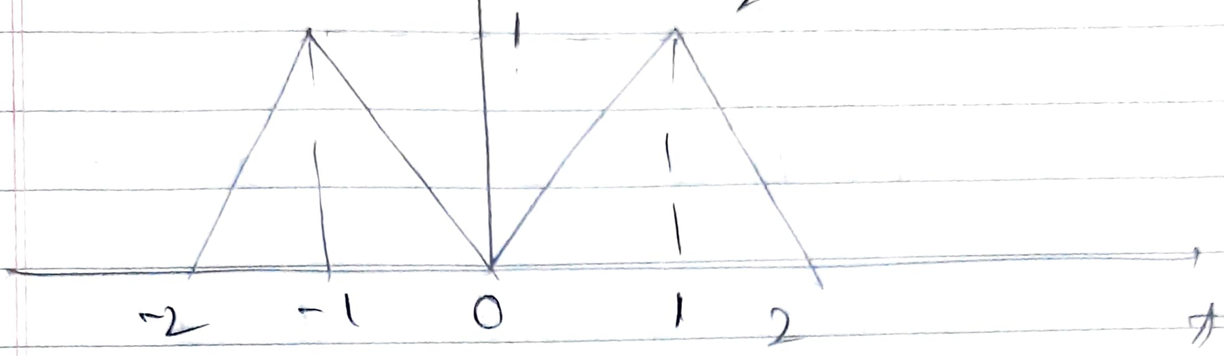


⑥

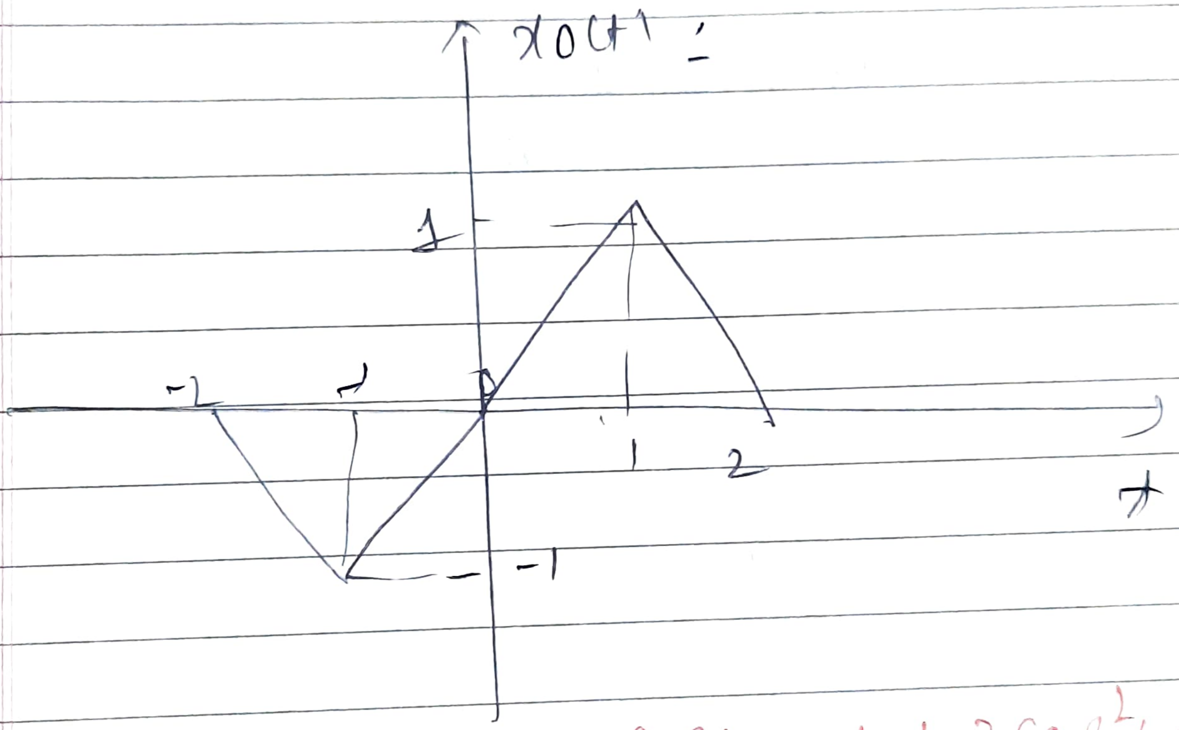




$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$



$x_o(t) =$



⑦ $x(t) = \cos\omega t + 2\sin\omega t + 3\cos^2\omega t$

$x(-t) = \cos\omega t - 2\sin\omega t + 3\cos^2\omega t$

$x_e(t) = \cos\omega t + 3\cos^2\omega t$

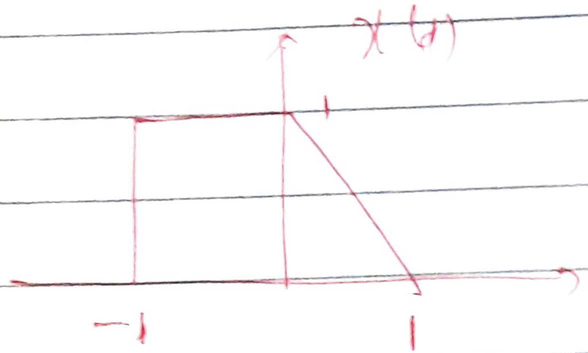
$x_o(t) = 2\sin\omega t$

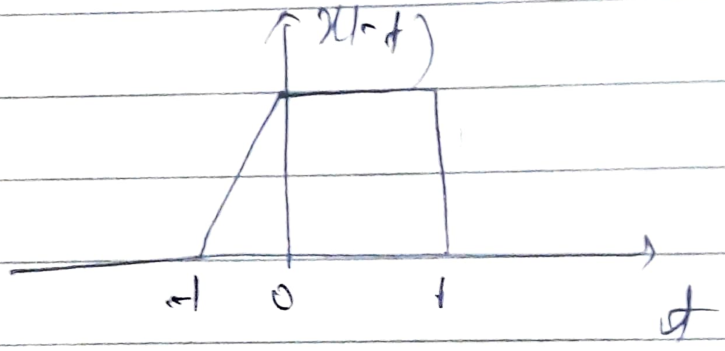
$x(t) = 1 + t\cos t + t^2\sin t$
$+ t^3 \sin t \cos t$

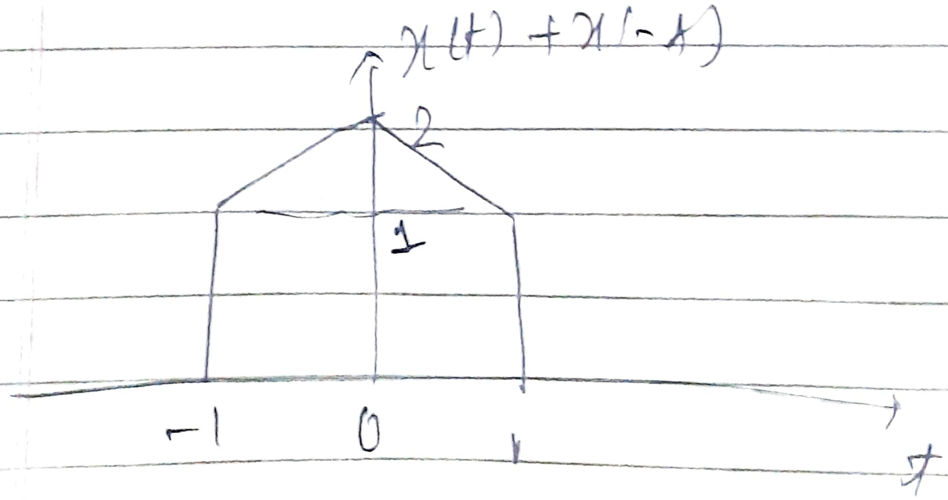
$x(-t) = 1 - t\cos t - t^2 \sin t + t^3 \sin t \cos t$

$x(t) + x(-t) = 2 + 2t^3 \sin t \cos t$

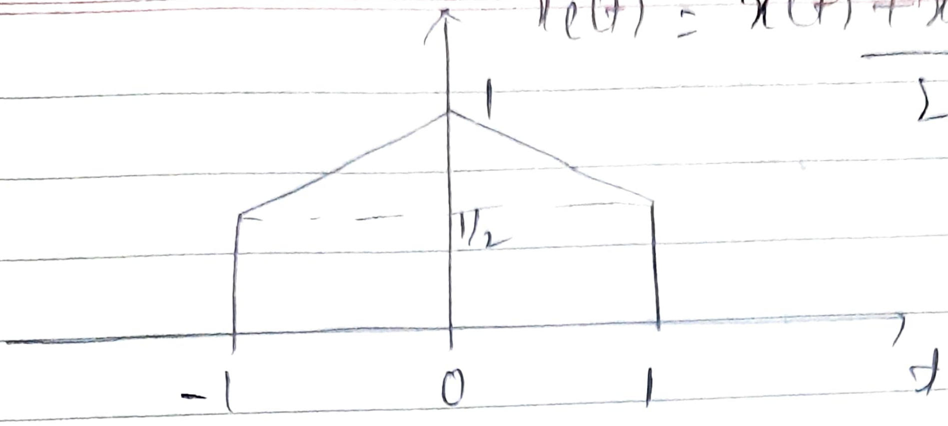
$x_e(t) = 1 + t^3 \sin t \cos t$
$x_o(t) = t\cos t + t^2 \sin t$

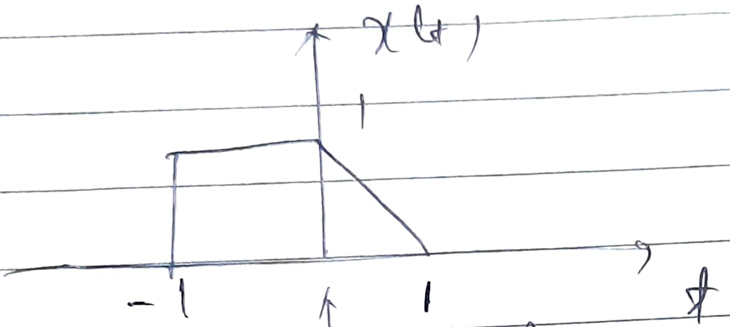


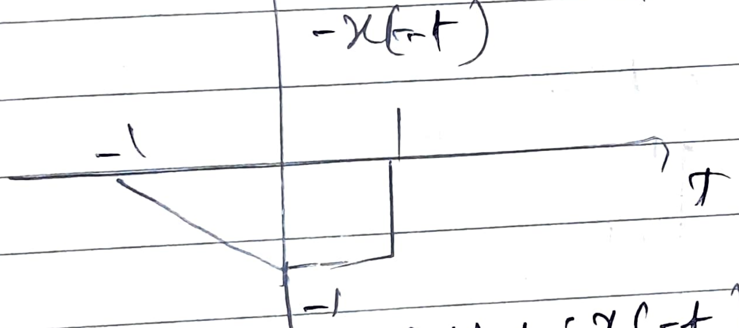


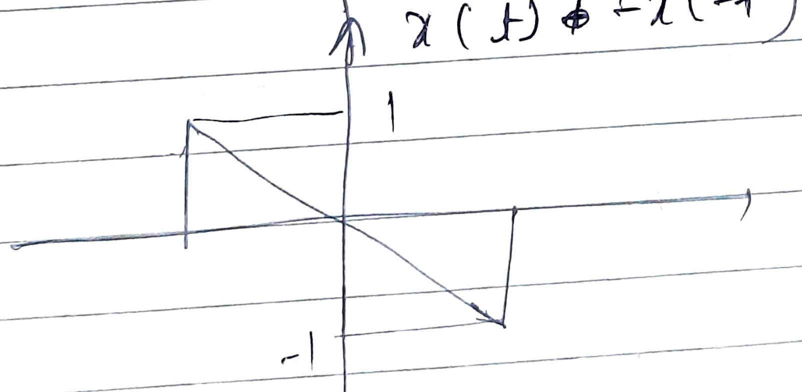


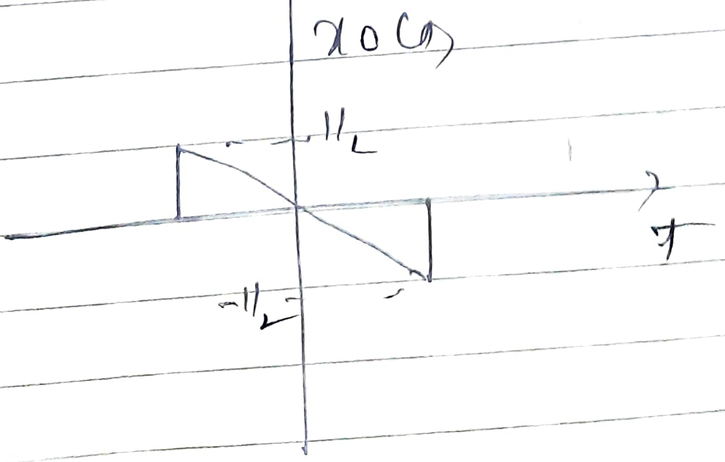
$$x_e(t) = \frac{x(t)+x(-t)}{2}$$











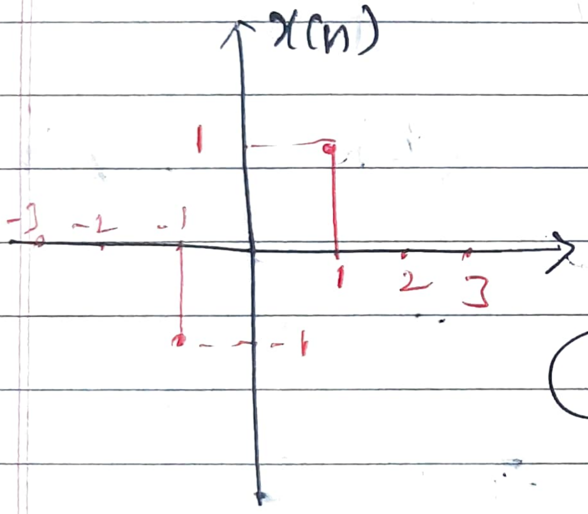
I Problems on operation of signals

① A discrete-time signal is defined by

$$x(n) = \begin{cases} 1, & n = 1 \\ -1, & n = -1 \\ 0, & n = 0 \ \& \ |n| > 1 \end{cases}$$

(T) Find $y(n)$ which is defined by

$y(n) = x(n) + x(-n)$



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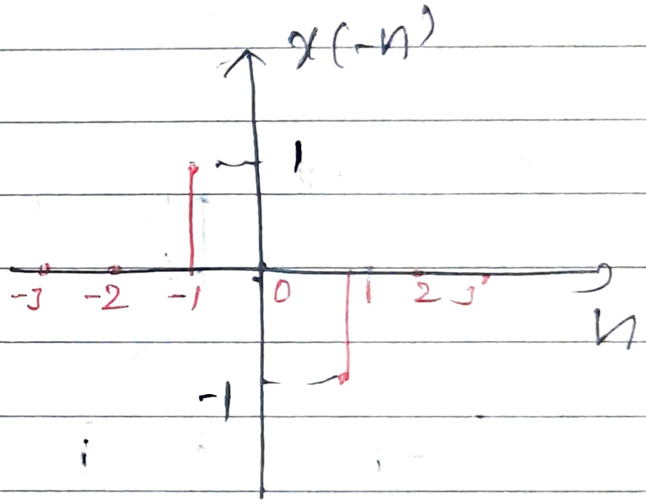
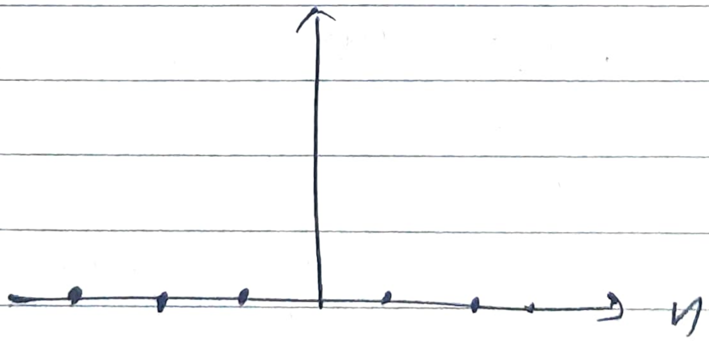




$y(n) = 0 \ \forall \ n$

② For signal $x(t)$ shown below sketch the foll signals

ⓐ $x(-t)$

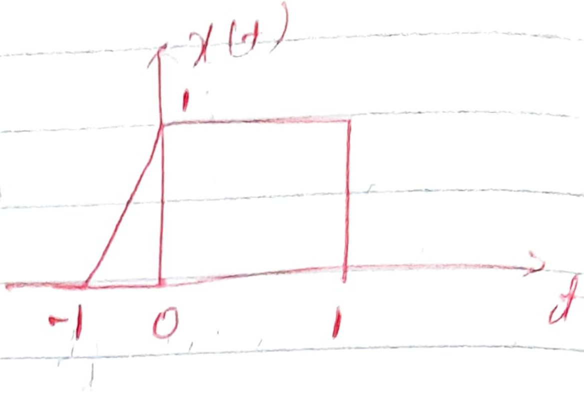
ⓑ $x(t+3)$

ⓒ $x(2t-2)$

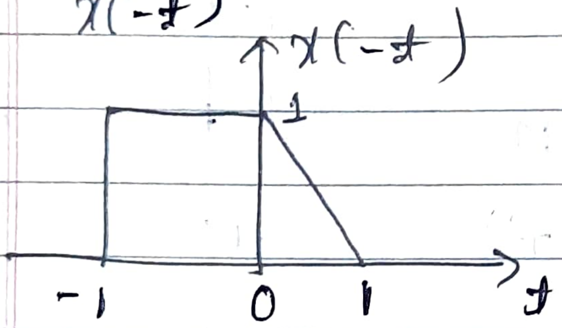
ⓓ $x(0.5t-1)$

ⓔ $x(2-2t)$

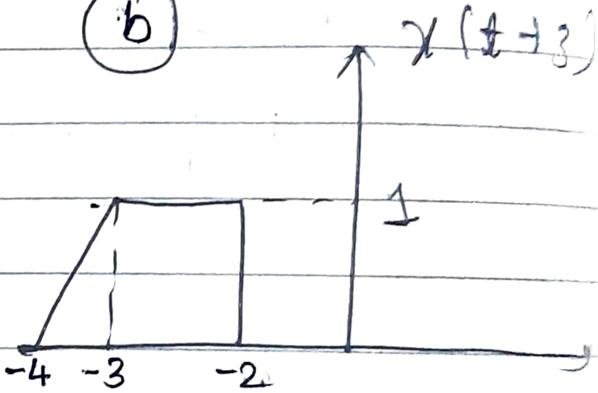
ⓖ $-x(3t+4)$ ⓕ $x(-t+3)$.



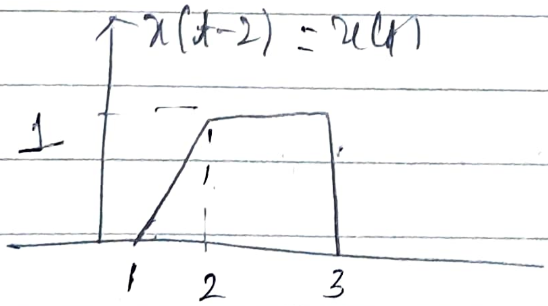
ⓐ $x(-t)$



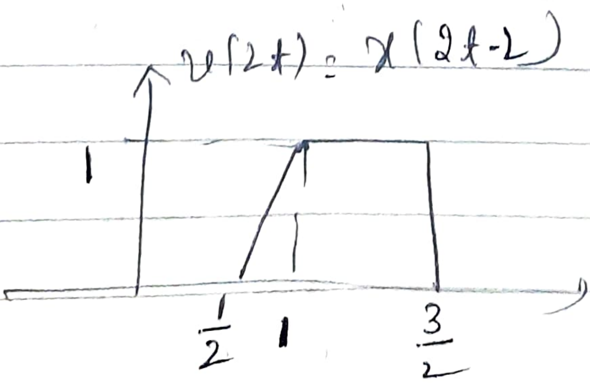
ⓑ



ⓒ $\underline{x(2t-2)}$ $x(t) \xrightarrow{\text{shift}} x(t-2) \xrightarrow{\text{scaling}} x(2t-2)$

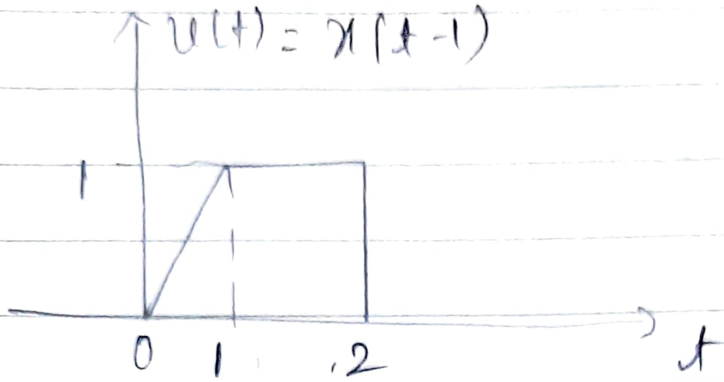


Then perform scaling on $v(t)$

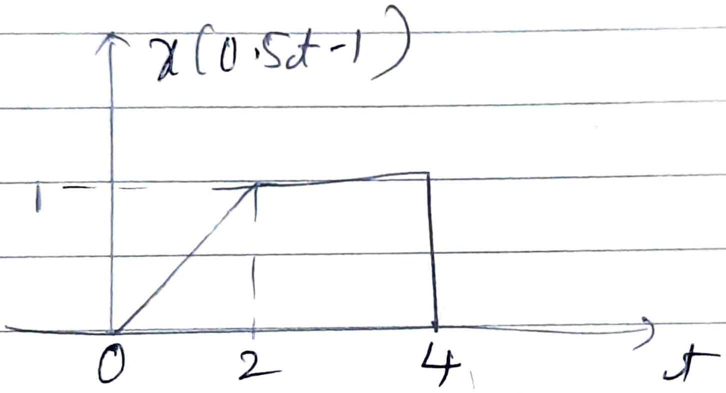


(d) $x(0.5t-1)$

(i) first perform $x(t-1) = v(t)$



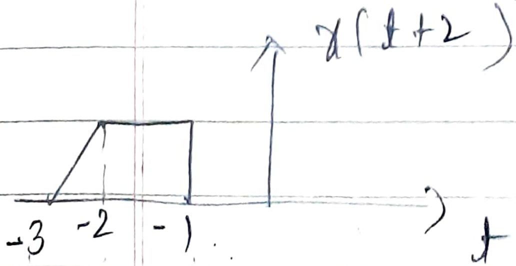
Then perform $v(0.5t) = x(0.5t-1)$

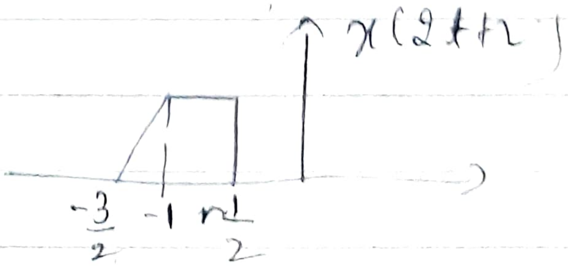


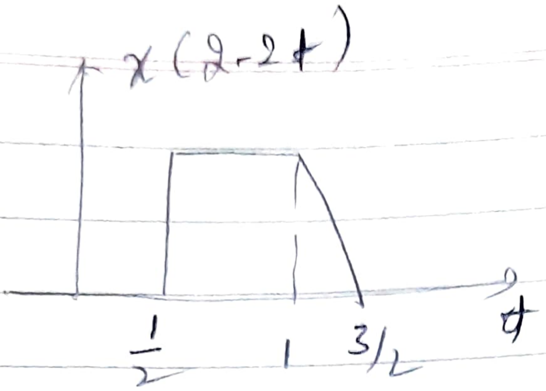
$x(-at+b)$

(e) $x(2-2t) = x(-2t+2)$ $x(-2t -$ [illegible]

(i) first perform $x(t+2) = v(t)$
(ii) Then perform $x(2t+2) = v(2t)$
(iii) Then perform $x(-2t+2) = v(-2t)$

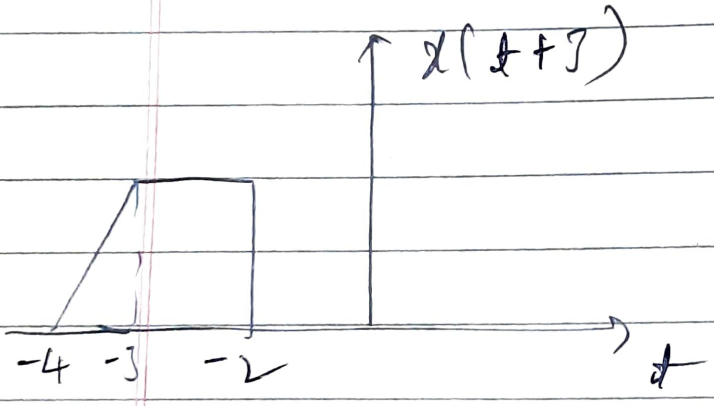


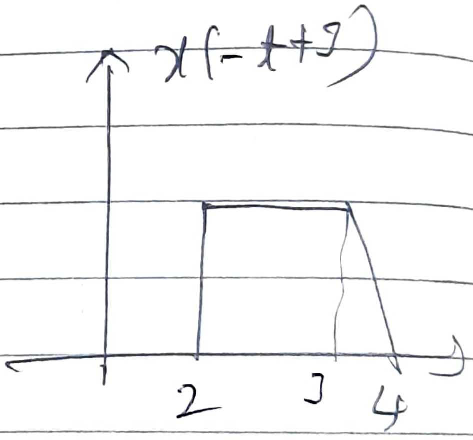




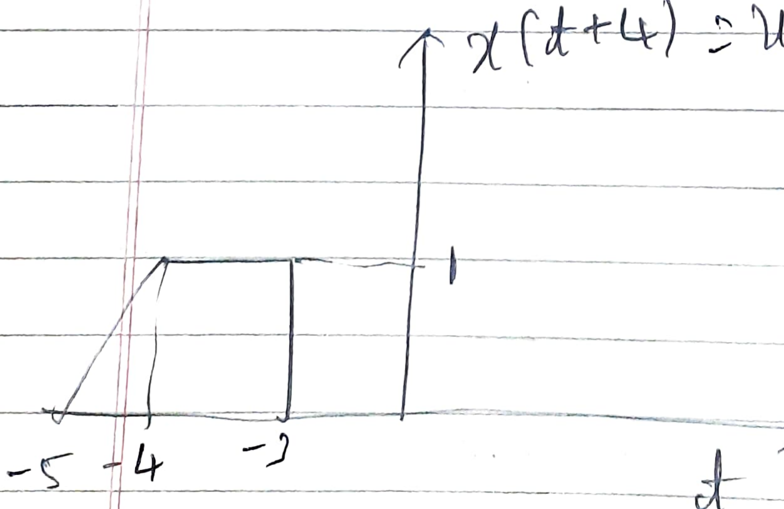
(f) $x(-t+3)$

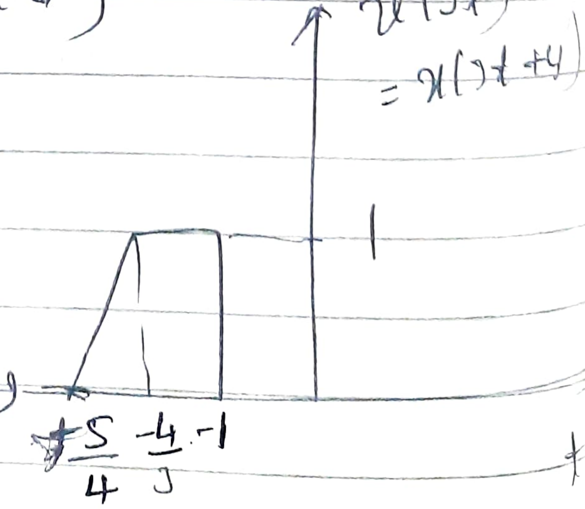
(i) first perform $x(t+3)$ then $x(-t)$





(g) $x(3t+4)$





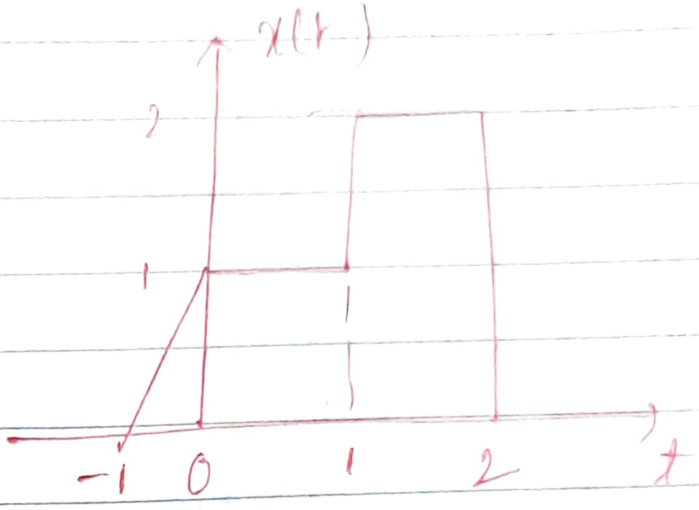
② Given $x(t)$ sketch the foll

ⓐ $x(-t)$

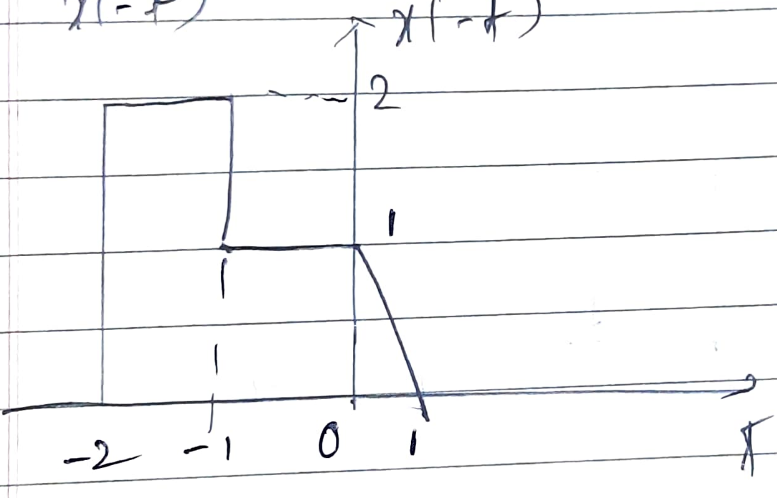
ⓑ $x(t-3/2)$

ⓒ $x(2t-1)$

ⓓ $x(4-t)$



ⓐ $x(-t)$



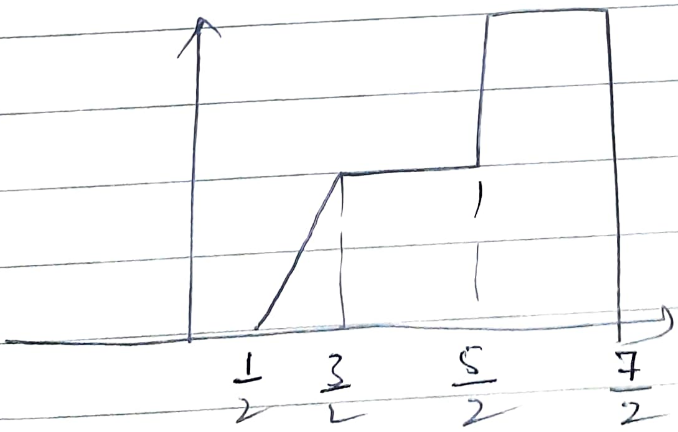
ⓑ $x(t-3/2)$

$-1 \rightarrow -1+\frac{3}{2} = \frac{-2+3}{2} = \frac{1}{2}$

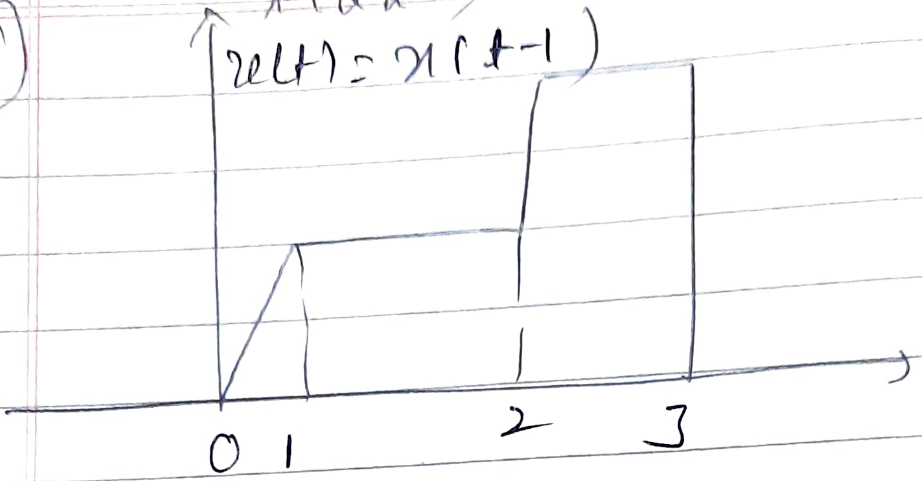
$1 \rightarrow 1+\frac{3}{2} = \frac{2+3}{2} = \frac{5}{2}$

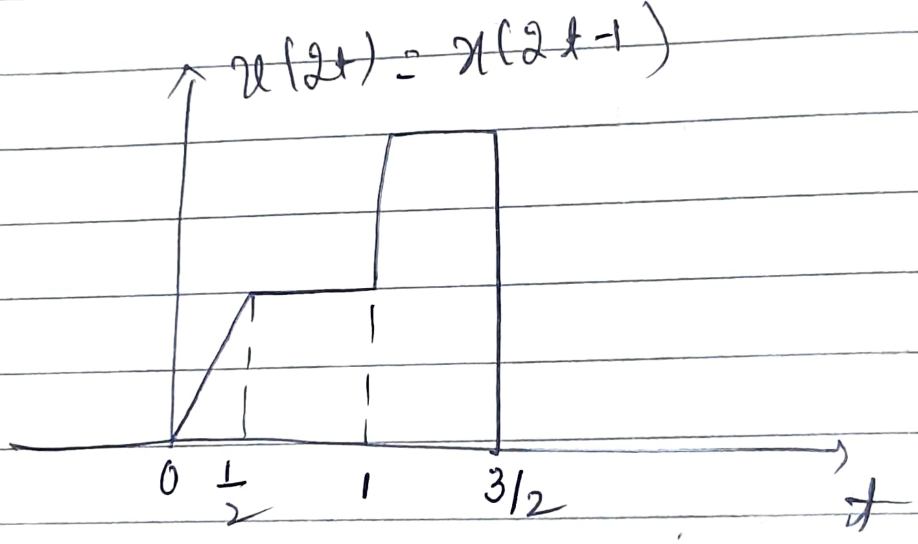
$2 \rightarrow 2+\frac{3}{2}$

$= \frac{4+3}{2} = \frac{7}{2}$

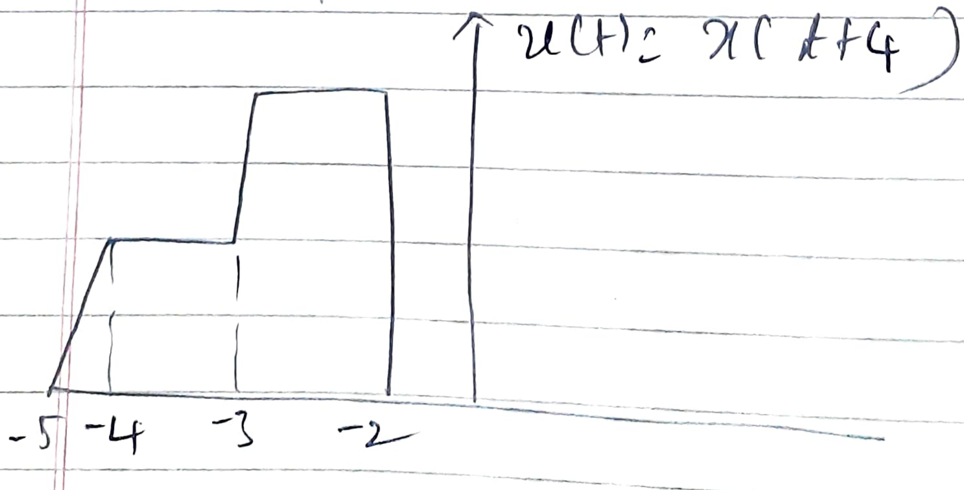


ⓒ $x(2t-1)$

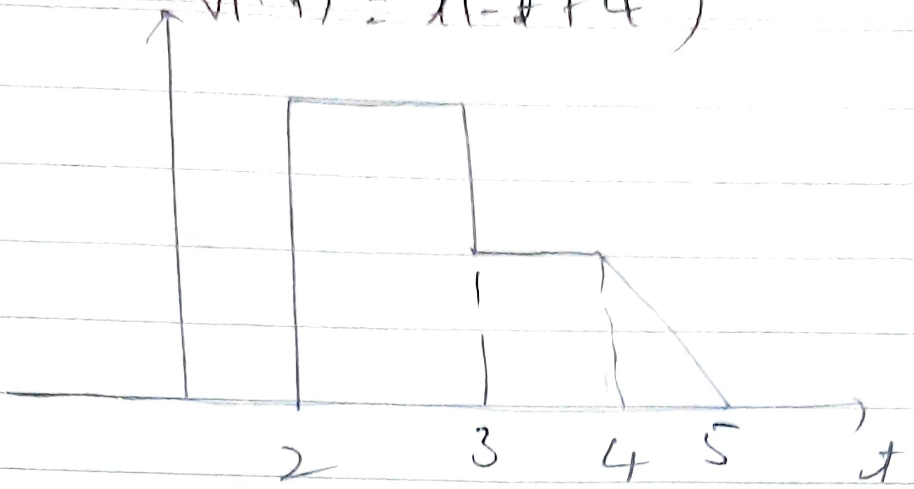




ⓓ $x(4-t) = x(-t+4)$

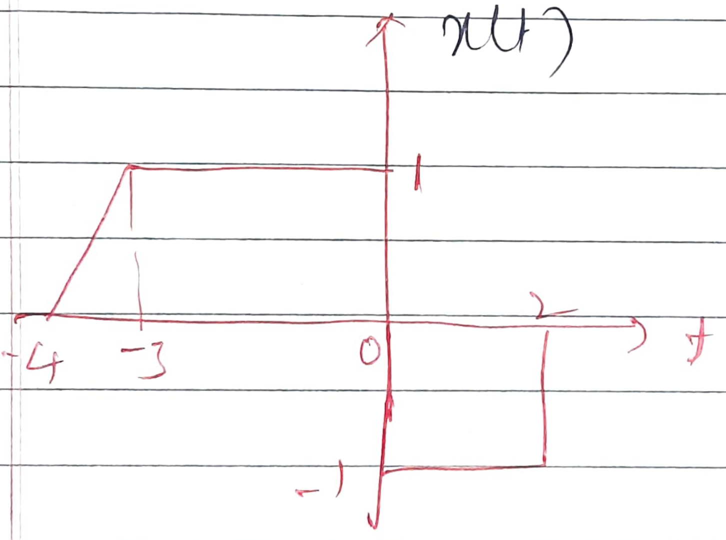


y(-t) = x(-t+4)

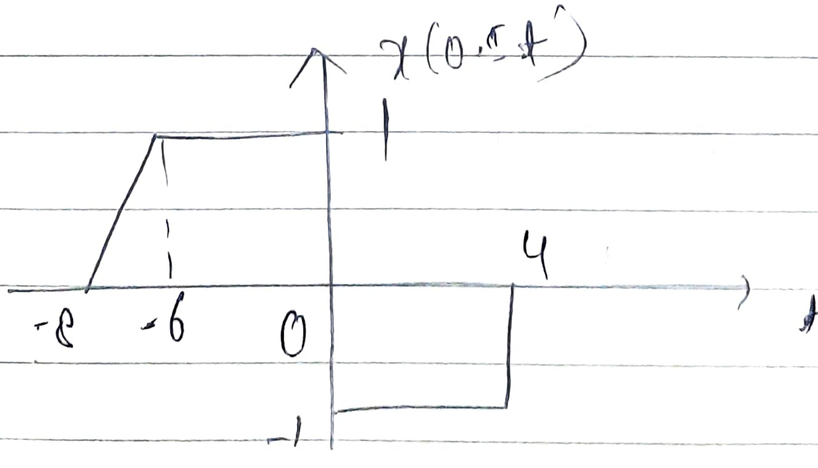


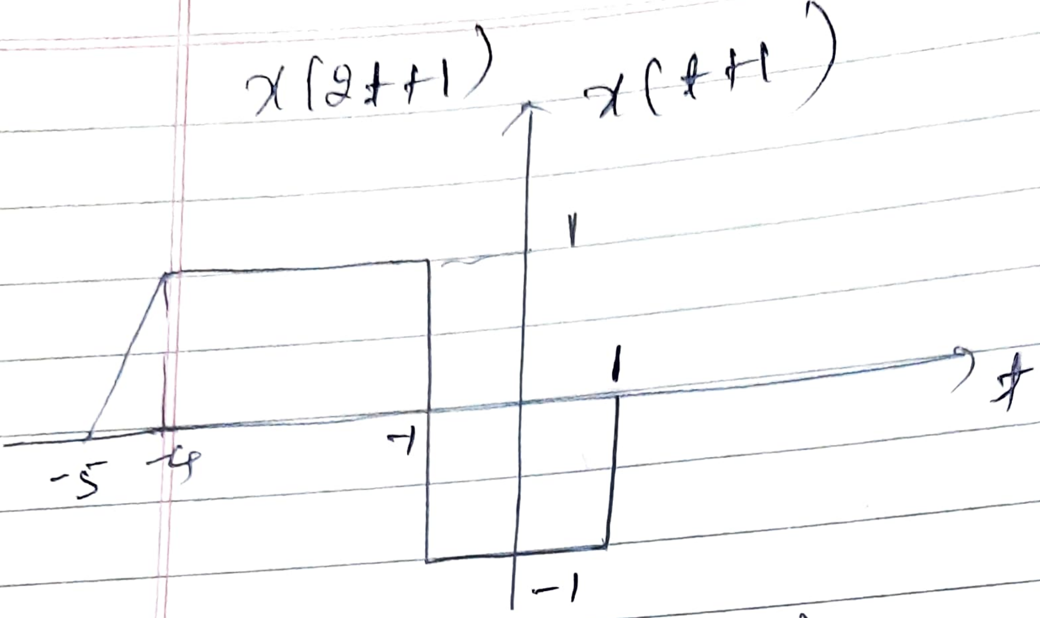
⑤ The signal x(t) is shown below sketch

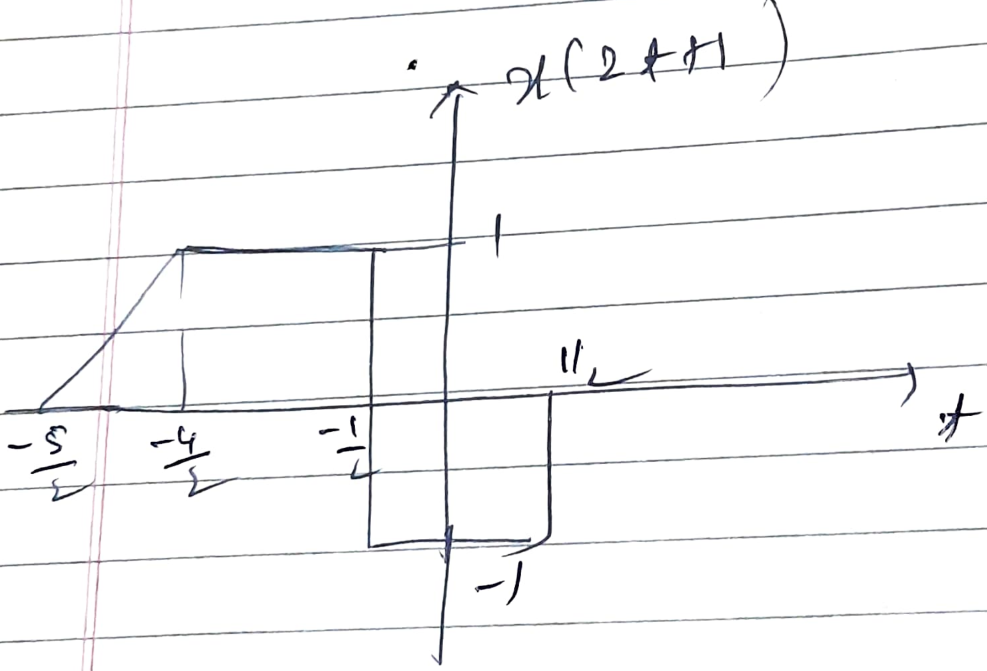
ⓐ x(0.5t) ⓑ x(2t+1) ⓒ x(3-t)
ⓓ x(t+5)



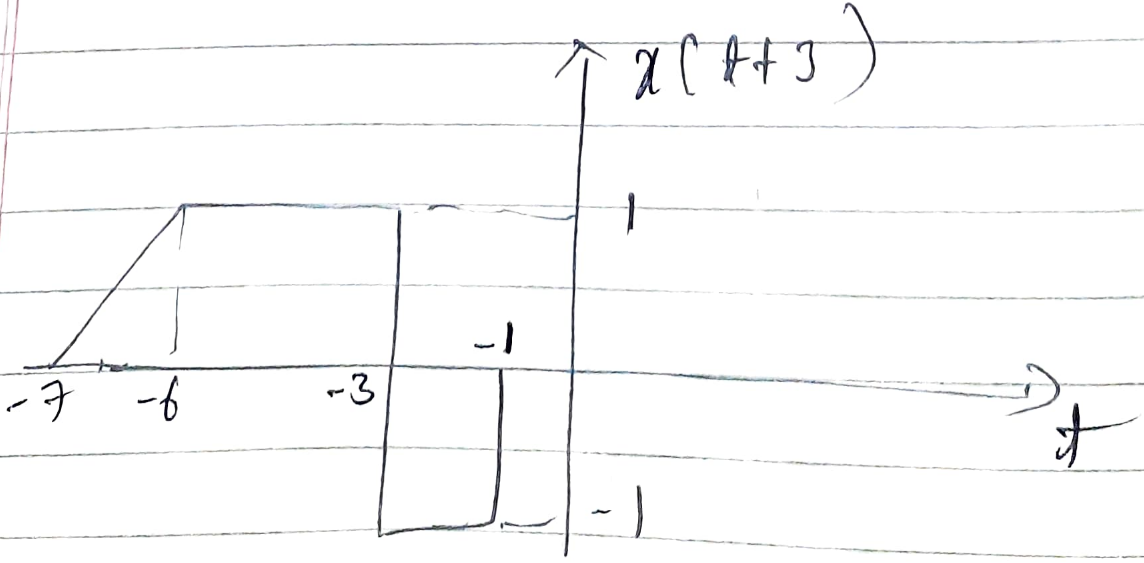
ⓐ

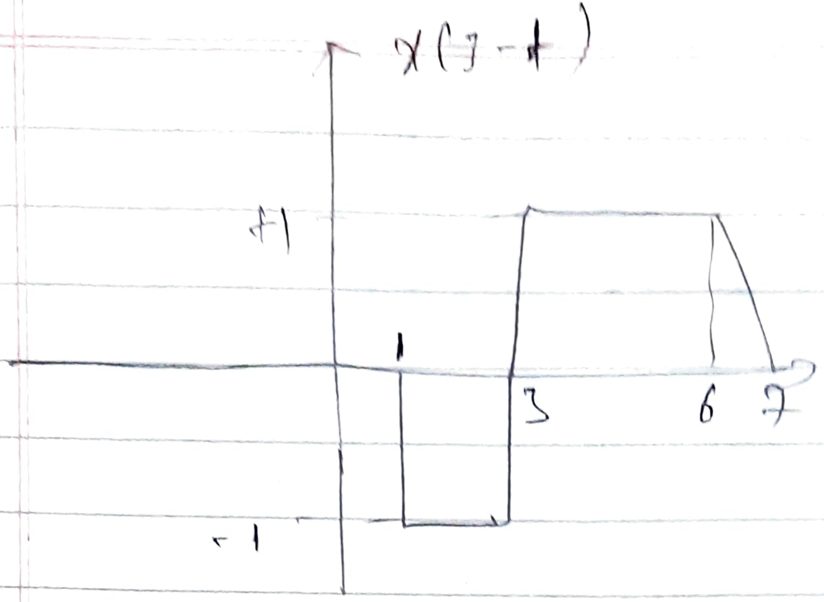



x(2t+1)
x(t+1)
1
1
-5
-4
-1
-1
t

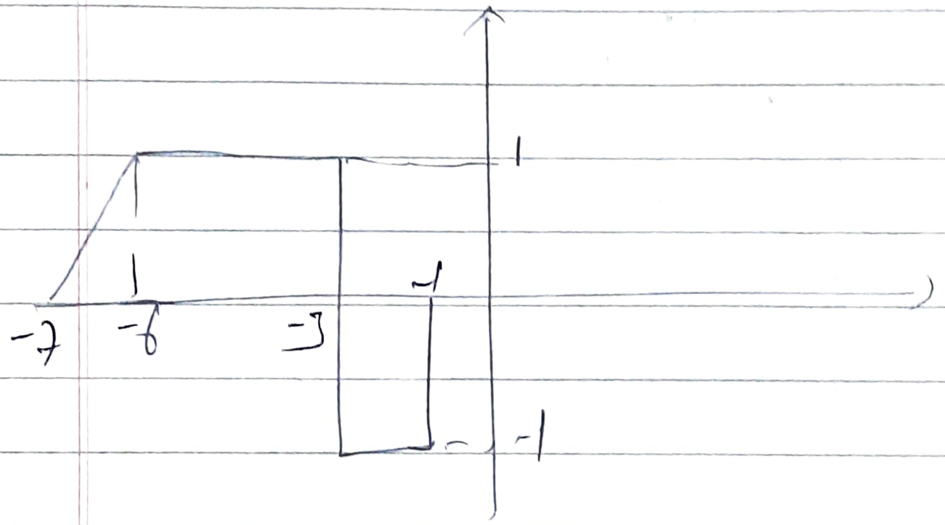

x(2t+1)
1
1/2
$-\frac{5}{2}$
$-\frac{4}{2}$
$-\frac{1}{2}$
-1
t

ⓒ $x(3-t) = x(-t+3)$


x(t+3)
1
-1
-7
-6
-3
-1
t



ⓓ $x(t+3)$



④ Given $x(n) = \{1, 4, 7, 9, 8\}$

$y(n) = \{3, 5, \underset{\uparrow}{0}, 6, 2\}$

Sketch the foll'

ⓐ $3x(n) + 2y(n)$

ⓑ $3x(n) - 2y(n)$

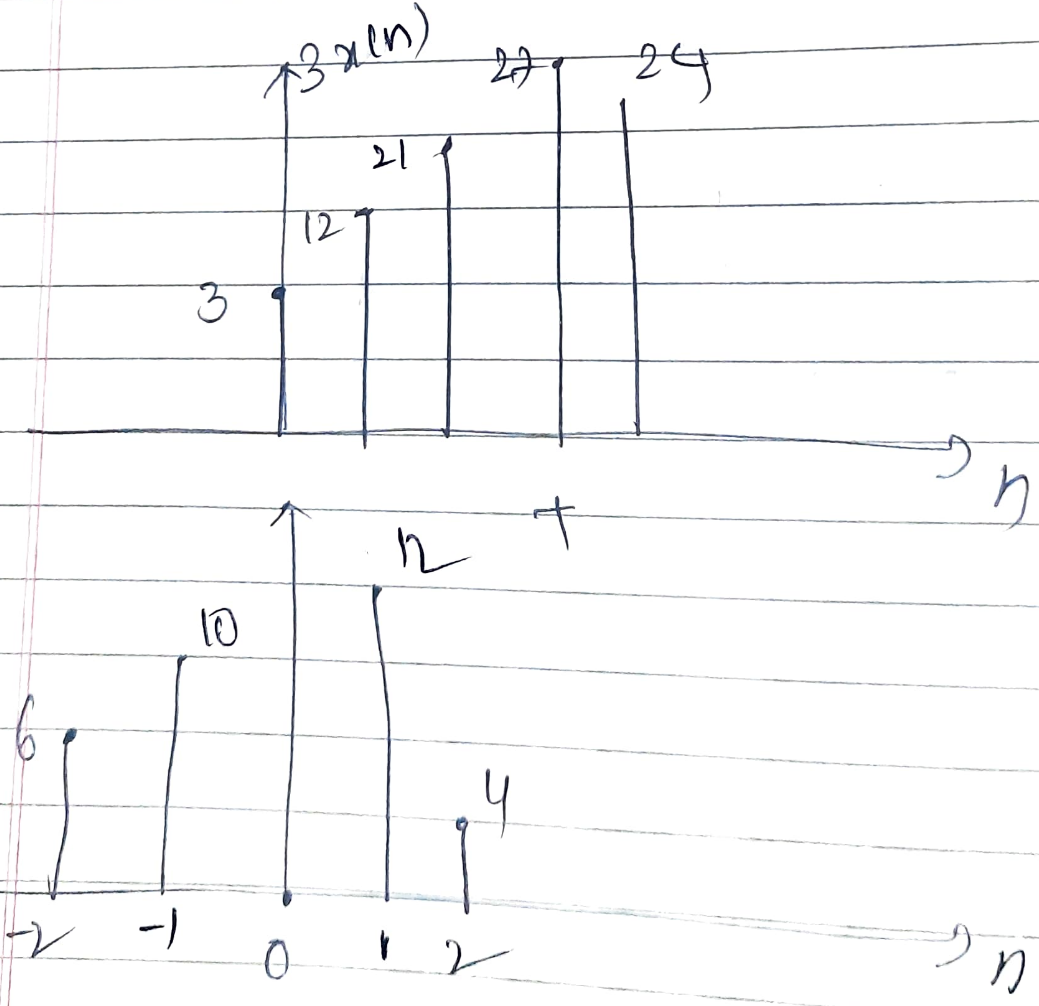
ⓒ $x(n) \cdot y(n-1)$

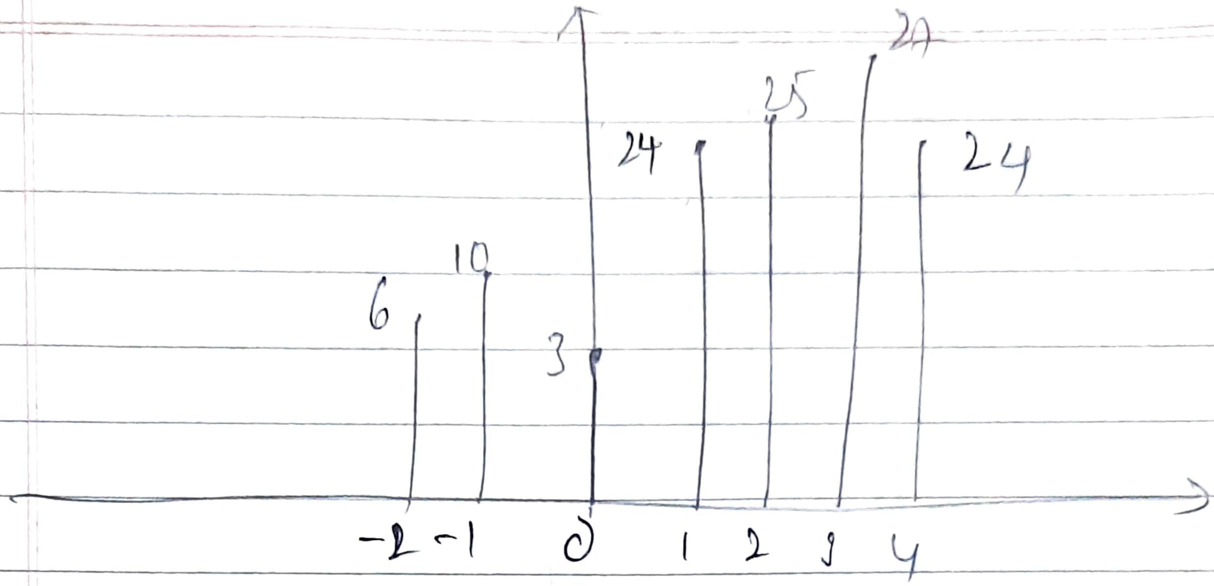
ⓐ $3x(n) + 2y(n)$

$3x(n) = \{3, 12, 21, 27, 24\}$

$2y(n) = \{6, 10, \underset{\uparrow}{0}, 12, 4\}$

$3x(n) + 2y(n)$

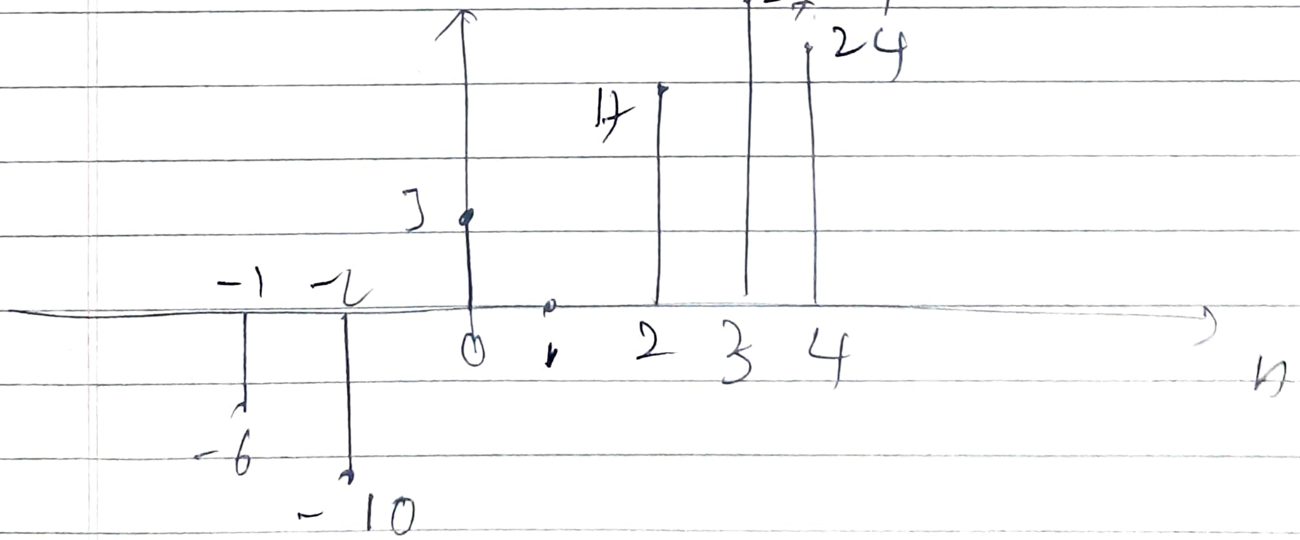




$\{6, 10, \underset{\uparrow}{3}, 24, 25, 27, 24\}$

ⓑ $3x(n) - 2y(n)$

$= \{-6, -10, \underset{\uparrow}{3}, 0, 17, 27, 24\}$

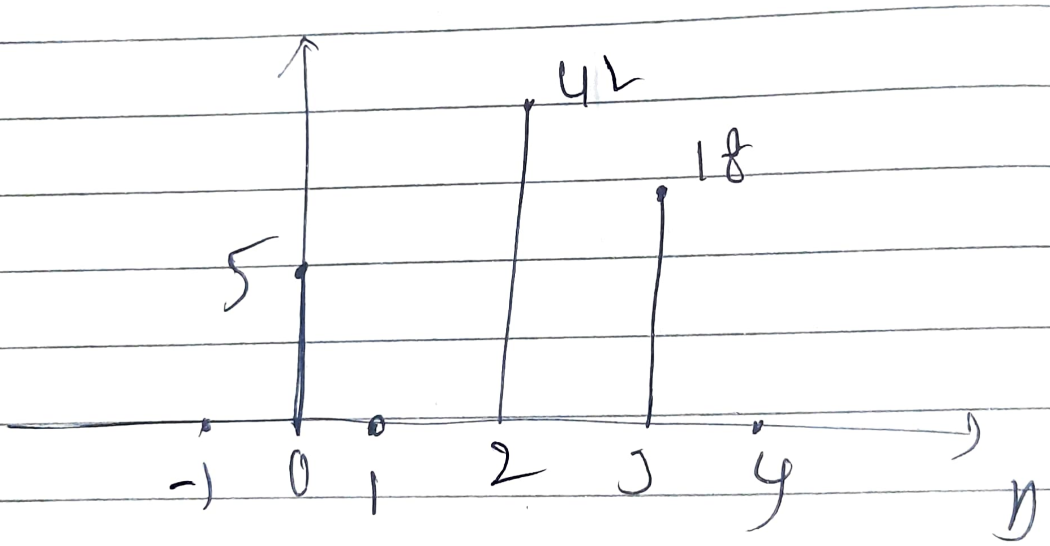


$x(n) \cdot y(n-1)$

$x(n) = \{1, 4, 7, 9, 8\}$

$y(n-1) : \{3, \underset{\uparrow}{5}, 0, 6, 2\}$

$(n-1) : \{0, \underset{\uparrow}{5}, 0, 42, 18, 0\}$



Given

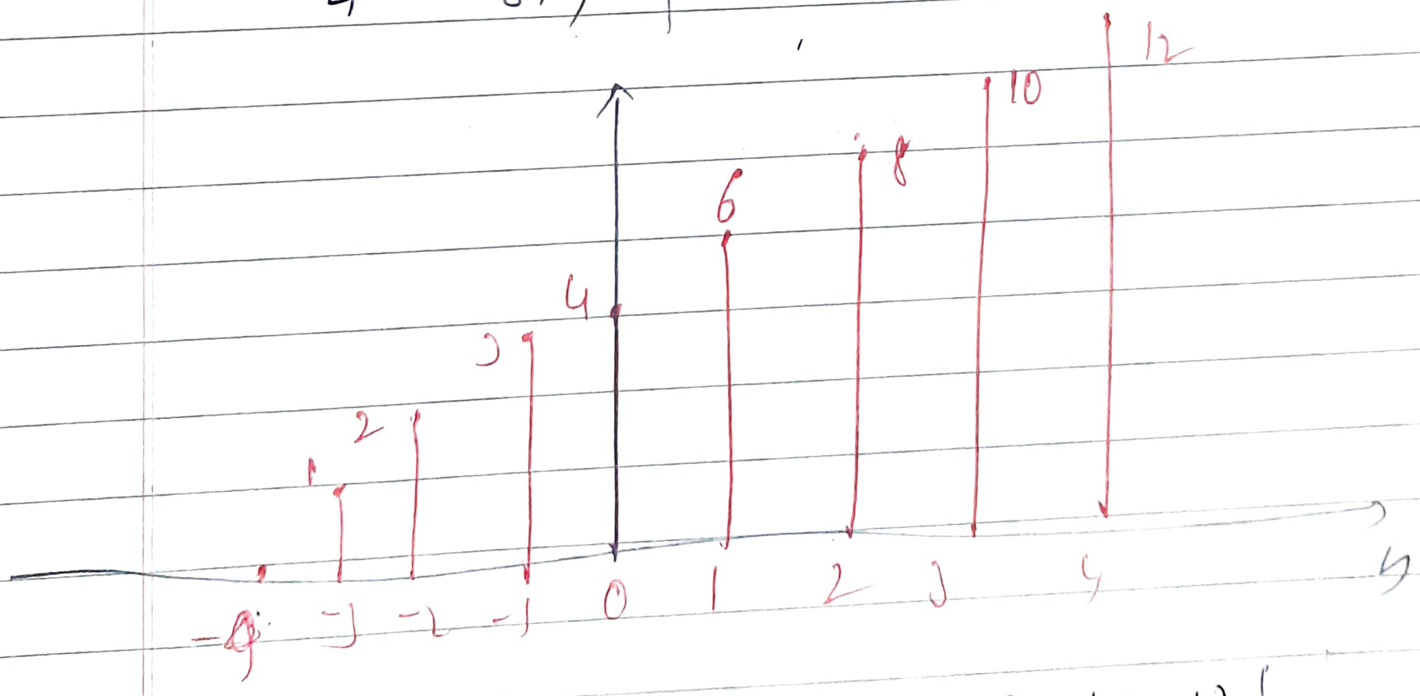
$$x(n) = \begin{cases} n+4, & -4 \leq n < -1 \\ 2n+4, & -1 \leq n \leq 4 \\ 0, & \text{elsewhere} \end{cases}$$

sketch

$y_1(n) = x(4-n)$

$y_2(n) = x(2n-3)$

n	$n+4$	$x(n)$
-4	$-4+4$	0
-3	$-3+4$	$+1$
-2	$-2+4$	2
-1	$-1+4$	3
	$2n+4$	
0	$0+4$	4
1	$2+4$	6
2	$4+4$	8
3	$6+4$	10
4	$8+4$	12

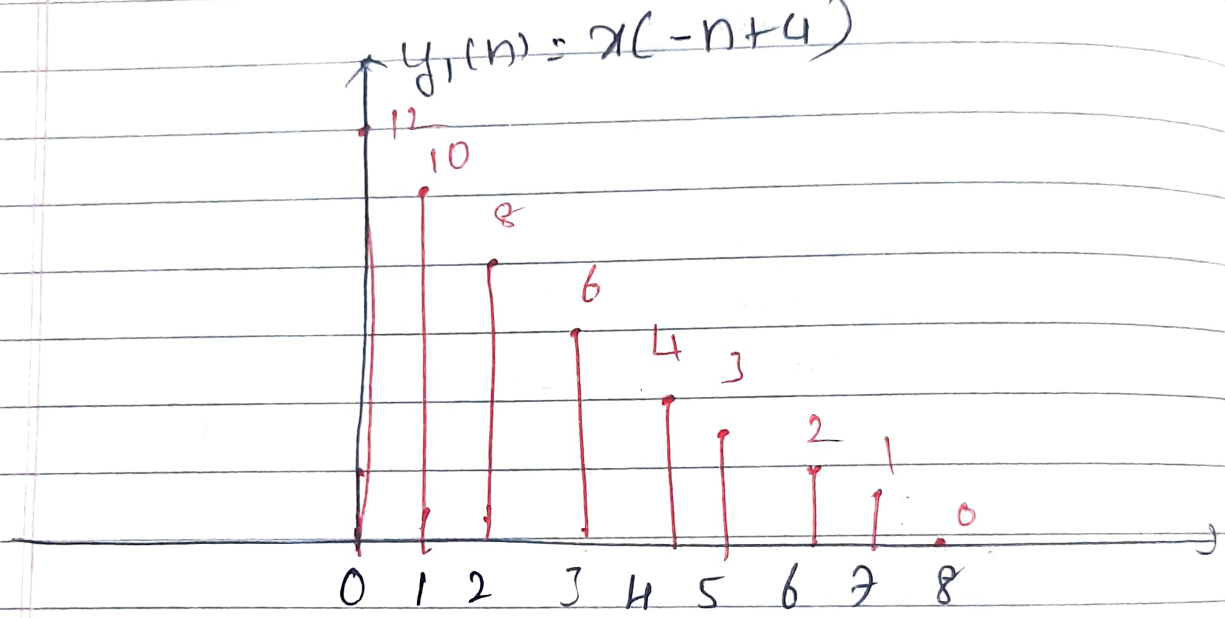


$x(n) = \{0, 1, 2, 3, \underset{\uparrow}{4}, 6, 8, 10, 12\}$

ⓐ $y_1(n) : x(4-n) = x(-n+4)$

$x(n) = x(n+4)$

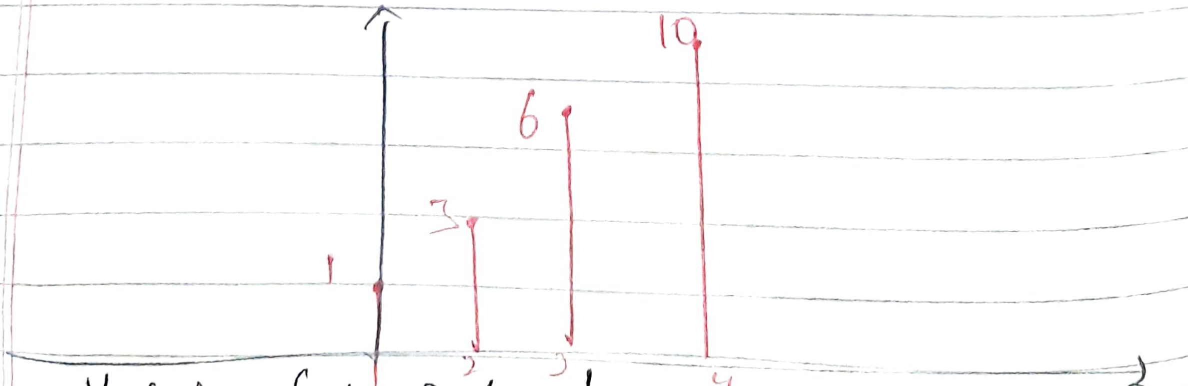
$$= \begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & 6 & 8 & 10 & 12 \\ -8 & -7 & -6 & -5 & -4 & -3 & -2 & -1 & \uparrow \end{array}$$



② $y_2(n) = x(2n-3)$

$x(n) = x(n-2) = \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 6 & 8 & 10 \\ -1 & \uparrow & 1 & 2 & 3 & 4 & 5 & 6 \end{array}$

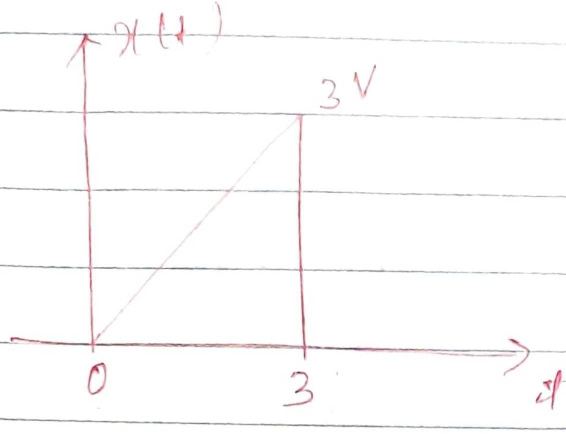
$y_2(n) = x(2n)$

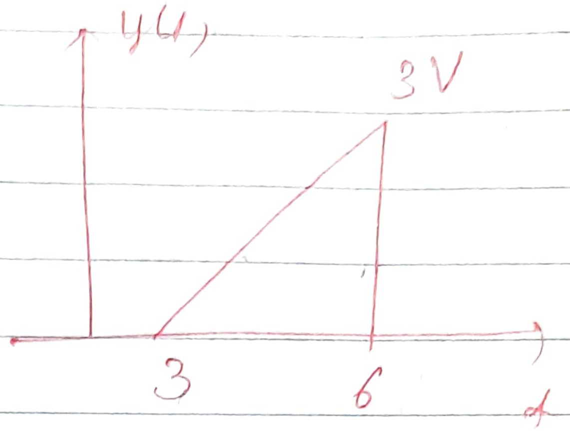


$y_2(n) = \{ \underset{\uparrow}{1}, 3, 6, 10 \}$

1 Express y(t) interms of x(t)

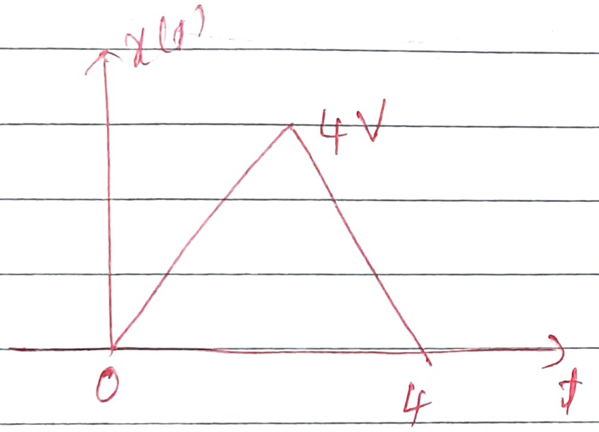
①

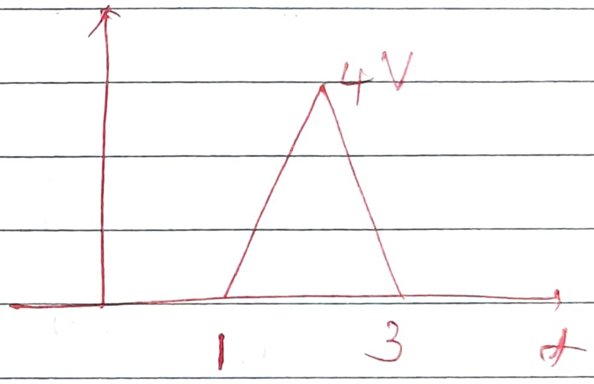


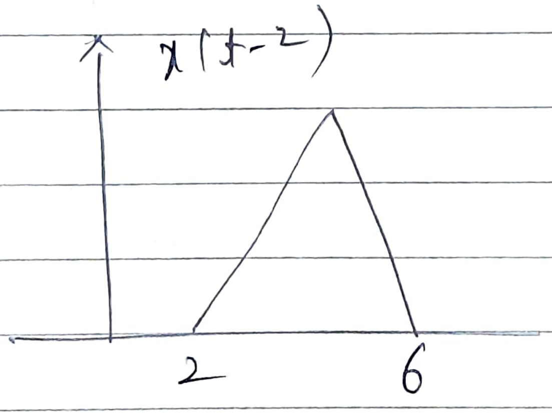


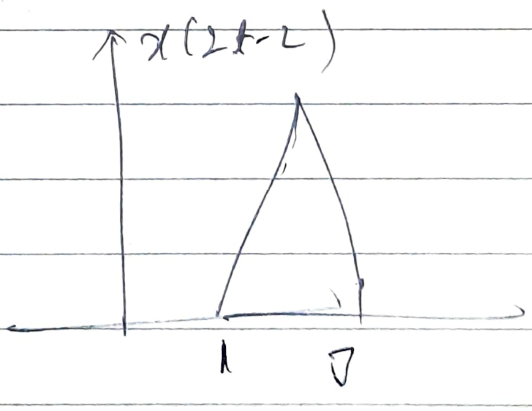
$y(t) = x(t-3)$

②



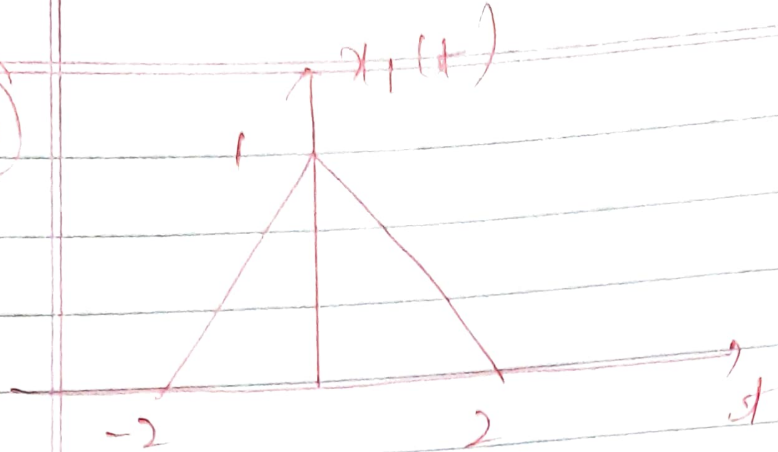


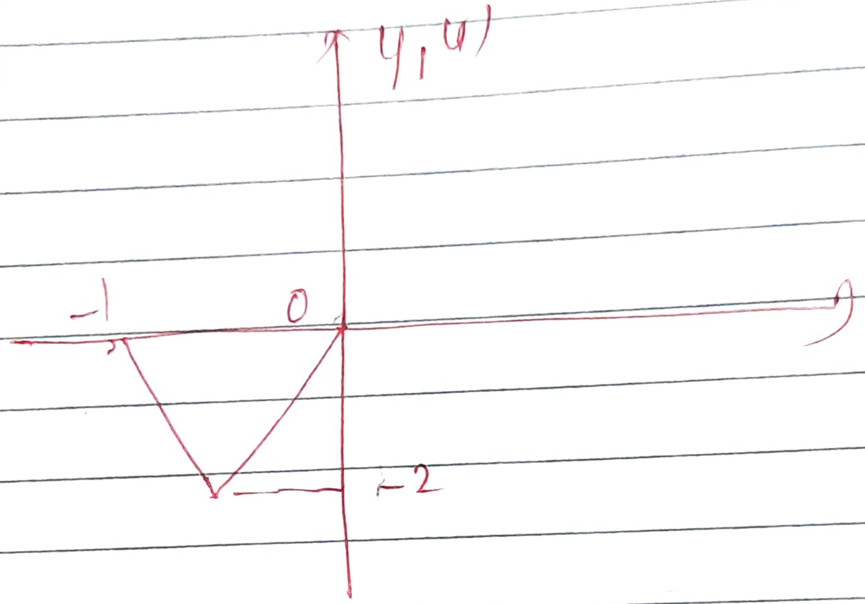


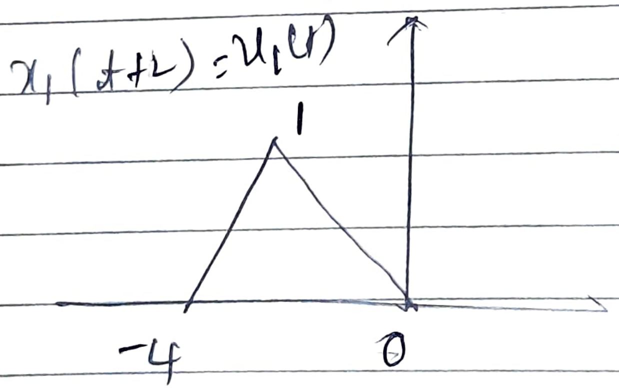


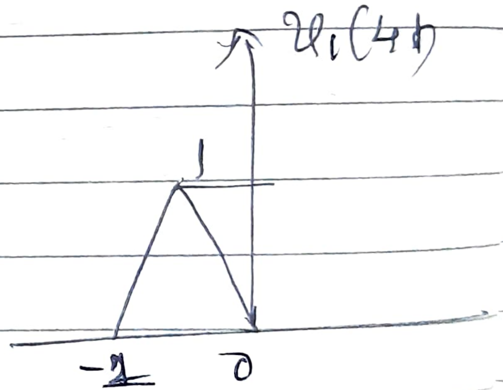
$y(t) = x(2t-2)$

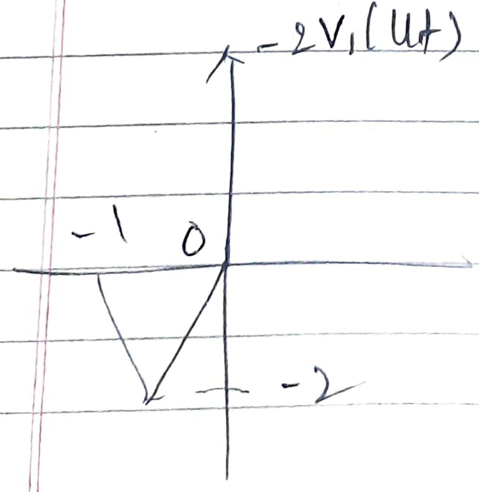
③





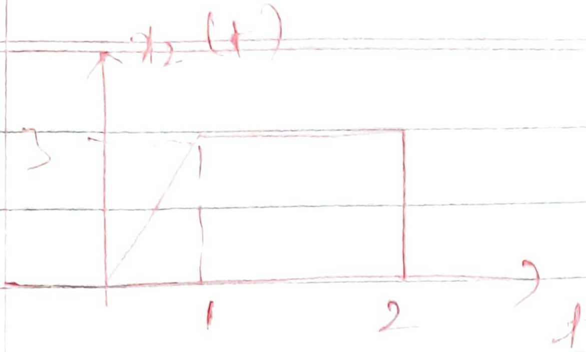


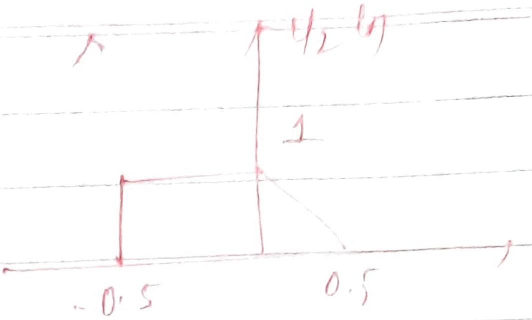


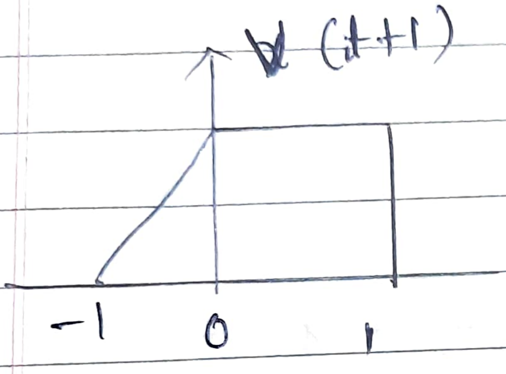


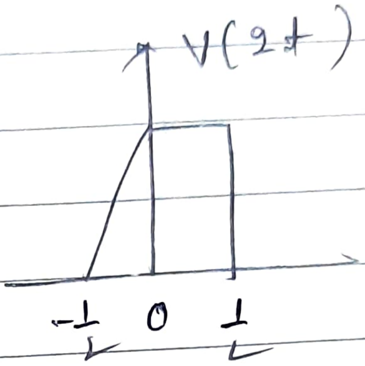
$$\boxed{y(t) = -2x_1(4t+2)}$$

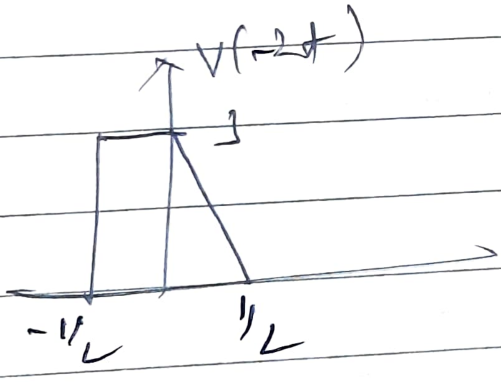
④

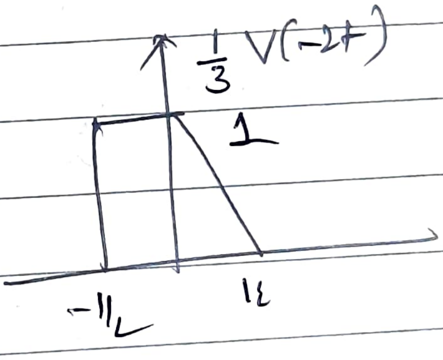












$$\boxed{y_2(t) = \frac{1}{3} x_2[-2t+1]}$$

II Problems on even & odd signals

① Find the even & odd parts of the foll signals

① $x(t) = e^{jt}$

$x(-t) = e^{-jt}$

WKT $x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right]$

& $x_0(t) = \frac{1}{2}\left[x(t) - x(-t)\right]$

$x_e(t) = \frac{1}{2}\left[e^{jt} + e^{-jt}\right]$

$= \underline{\cos t}$

$x_0(t) = \frac{1}{2}\left[e^{jt} - e^{-jt}\right]$

$= \underline{j \sin t}$

$\cos x = \frac{e^{jx} + e^{-j}}{2}$

$\sin x = \frac{e^{jx} - e^{-j}}{2j}$

① $x(t) = 4 + 3t + 5t^2 + 8t^3 + 9t^4$

$x(-t) = 4 - 3t + 5t^2 - 8t^3 + 9t^4$

$[x(t) + x(-t)] = 8 + 10t^2 + 18t^4$

$x_e(t) = \frac{x(t) + x(-t)}{2}$

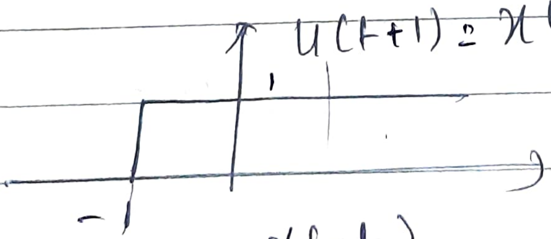
$= 4 + 5t^2 + 9t^4$

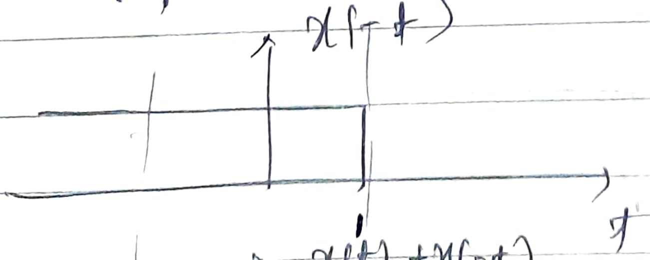
$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$

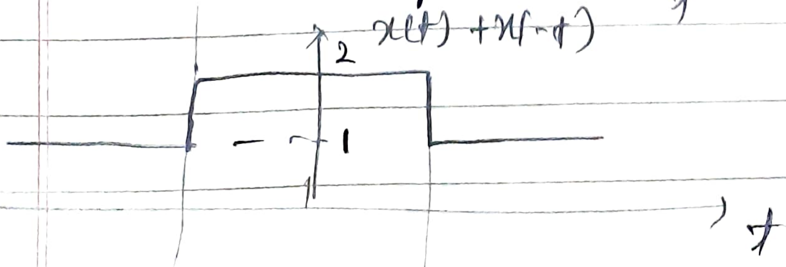
$= \underline{\underline{3t + 8t^3}}$

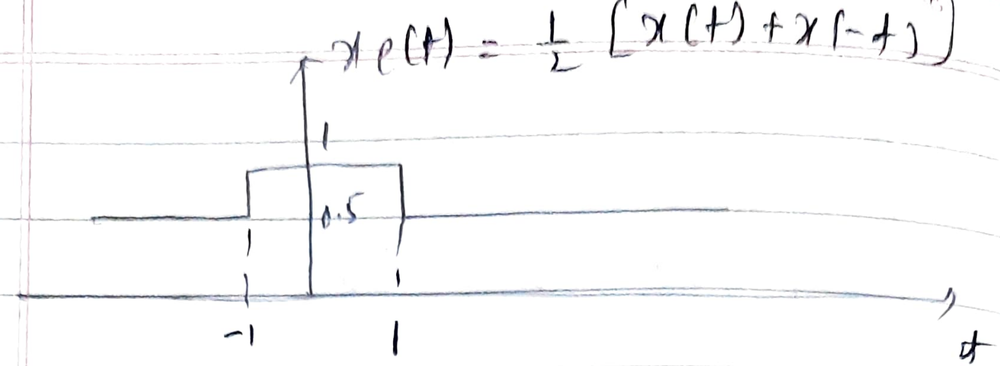
③ $x(t) = u(t+1)$

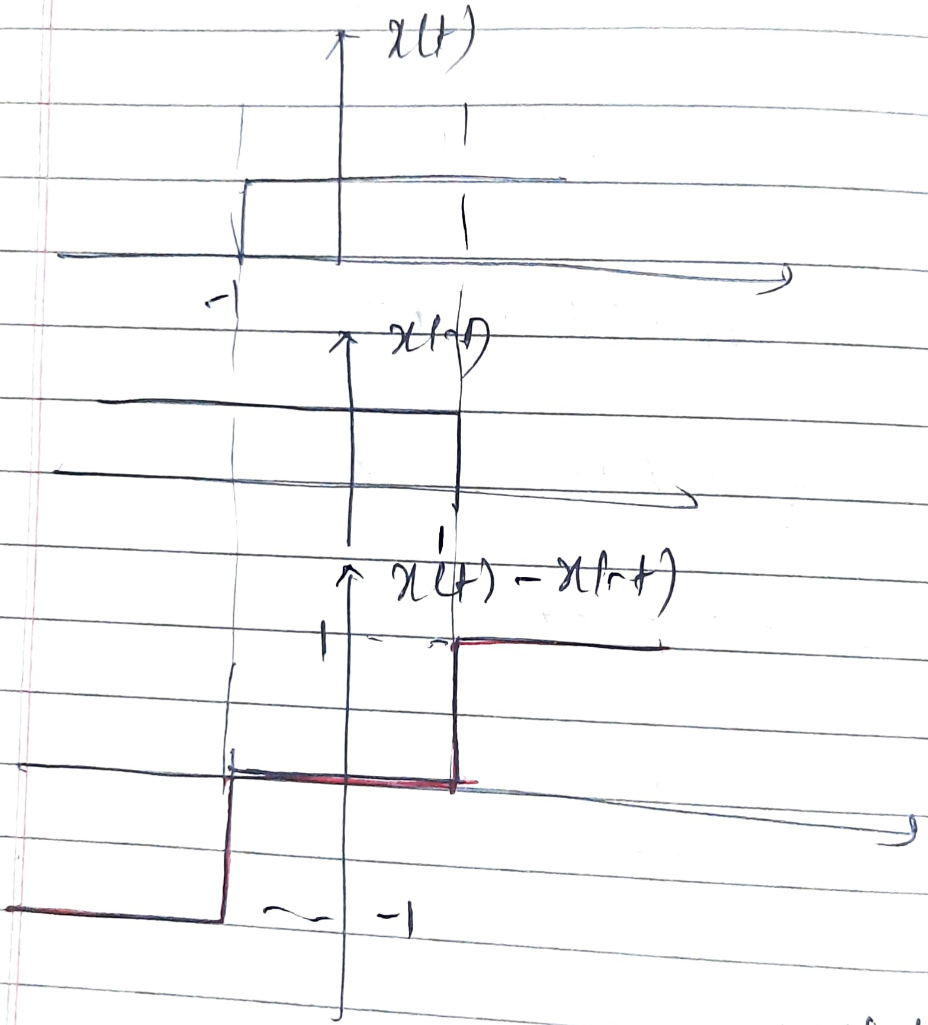
$x(-t) = u(-t+1)$

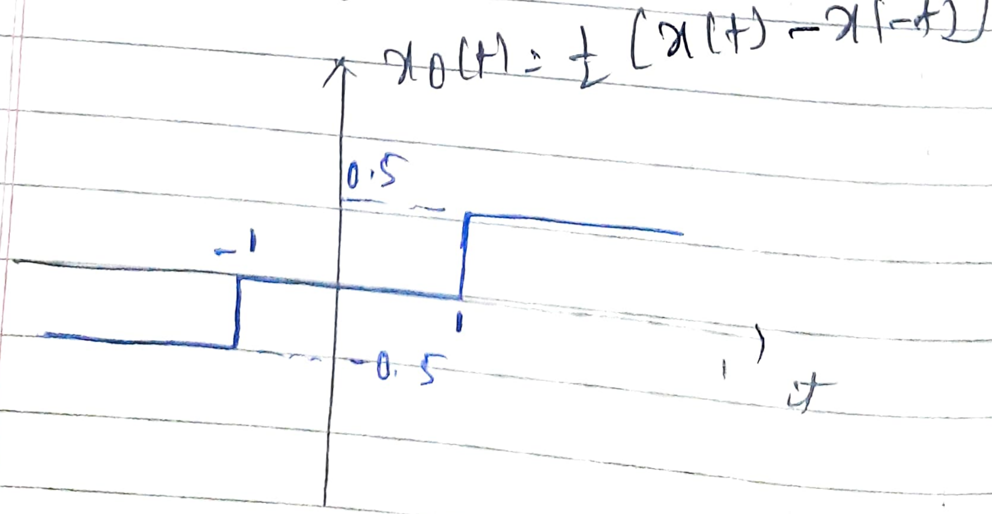







xe(t) = 1/2 [x(t) + x(-t)]
1
0.5
-1
1
t


x(t)
1
-1
x(-t)
1
x(t) - x(-t)
1
-1


x0(t) = 1/2 (x(t) - x(-t))
0.5
-1
1
-0.5
t

④ $x(n) = \{-1, \underset{\uparrow}{2}, -3, 4, -5\}$

$x(-n) = \{-5, 4, -3, \underset{\uparrow}{2}, -1\}$

$x(n) + x(-n) = -5, 4, -4, 4, -4, 4, -5$

$x_e(n) = \{-\frac{5}{2}, 2, -2, 2, -2, 2, -\frac{5}{2}\}$

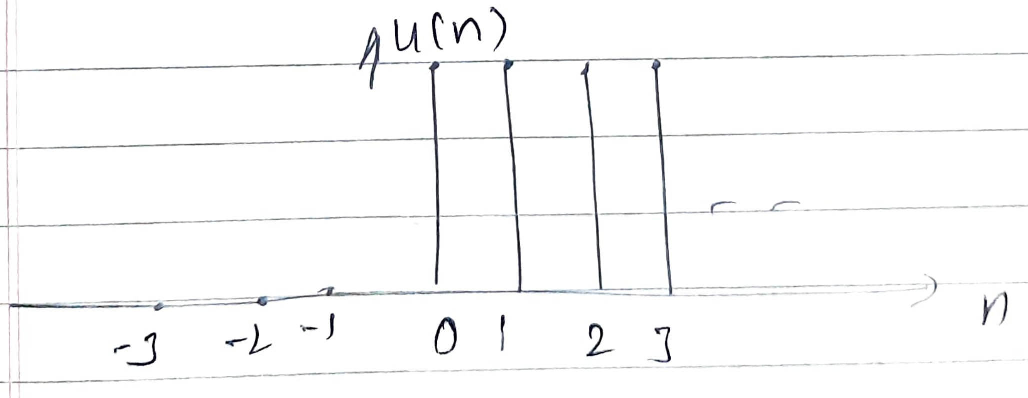
$x(n) - x(-n)$

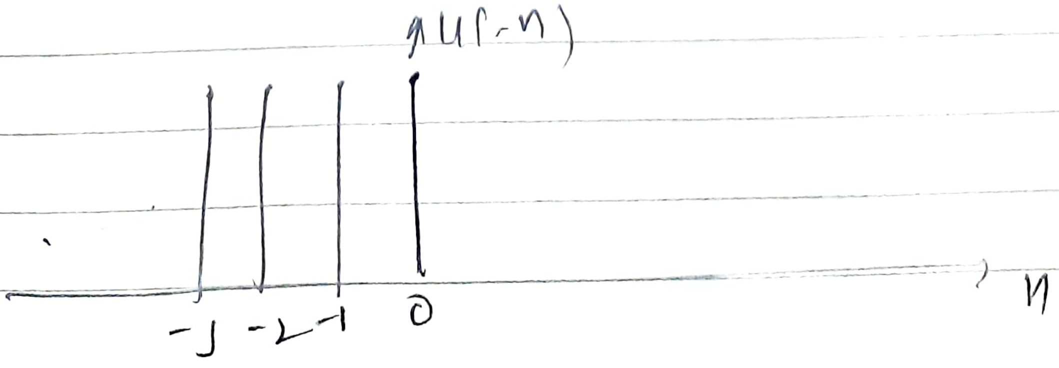
$= \{5, -4, 2, \underset{\uparrow}{0}, -2, 4, -5\}$

$x_o(n) = \{\frac{5}{2}, -2, 1, 0, -1, 2, -\frac{5}{2}\}$

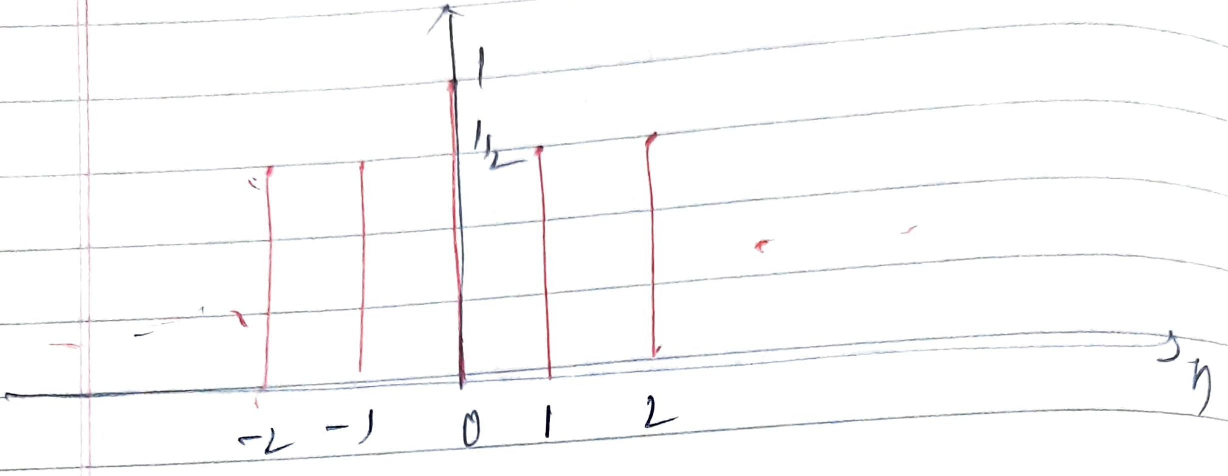
⑤ $x(n) = u(n)$

$x(-n) = u(-n)$

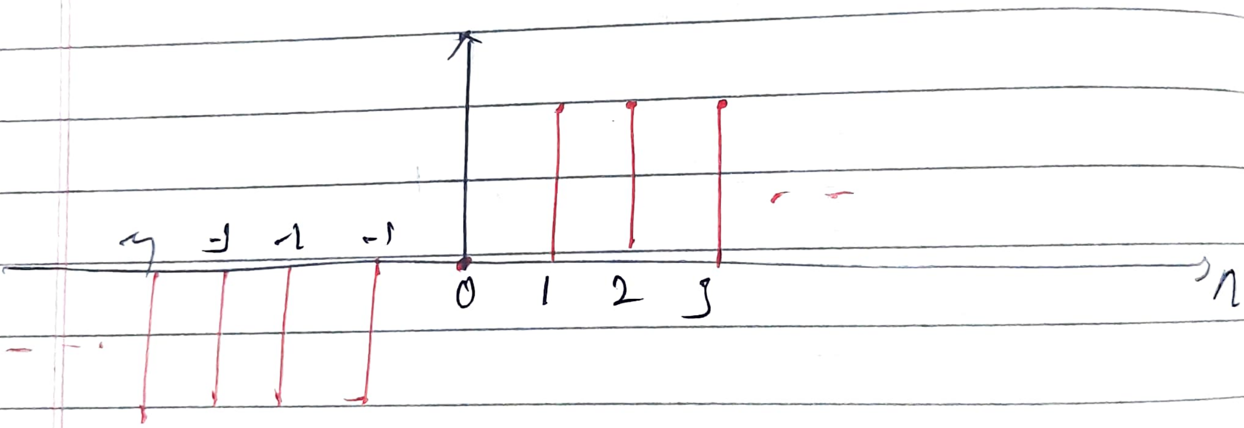




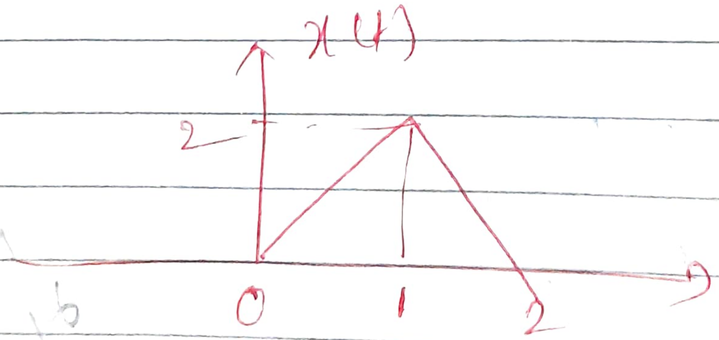
$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$

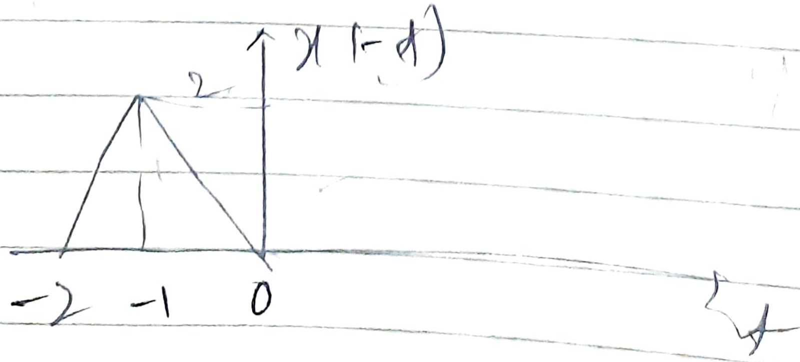


$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$

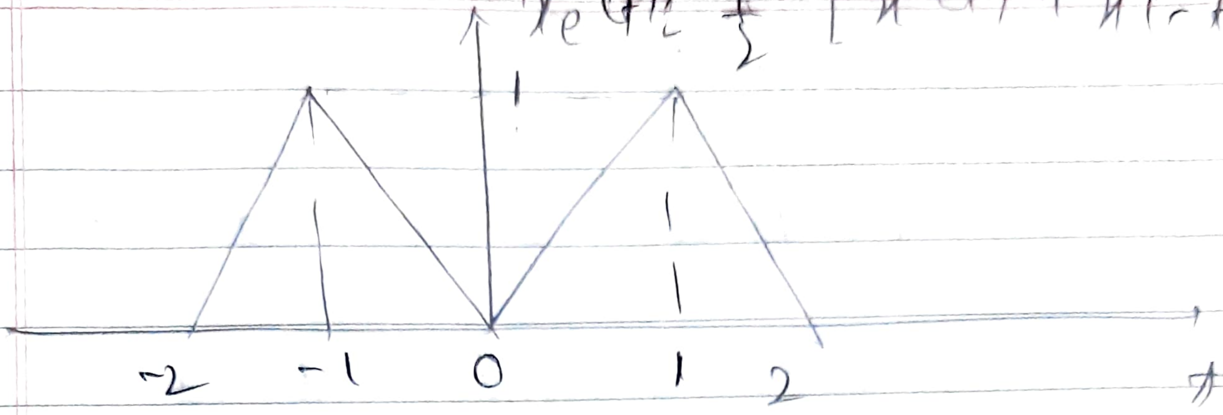


⑥

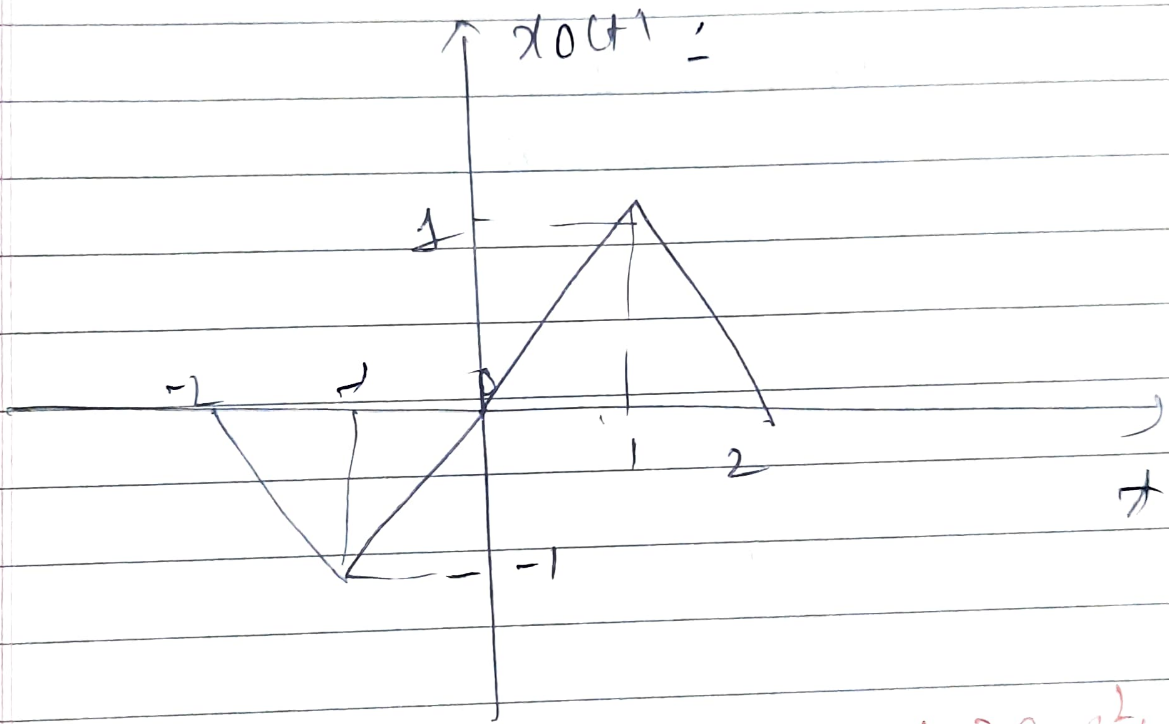




$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$



$x_o(t) =$



⑦ $x(t) = \cos\omega t + 2\sin\omega t + 3\cos^2\omega t$

$x(-t) = \cos\omega t - 2\sin\omega t + 3\cos^2\omega t$

$x_e(t) = \cos\omega t + 3\cos^2\omega t$

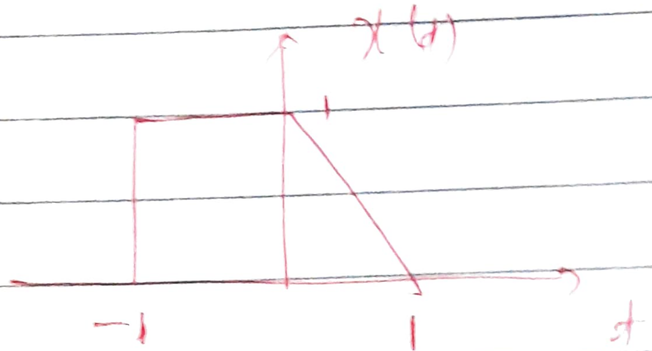
$x_o(t) = 2\sin\omega t$

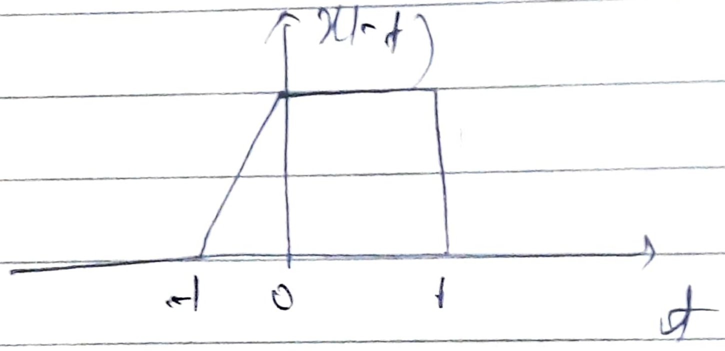
$x(t) = 1 + t\cos t + t^2\sin t$
$+ t^3 \sin t \cos t$

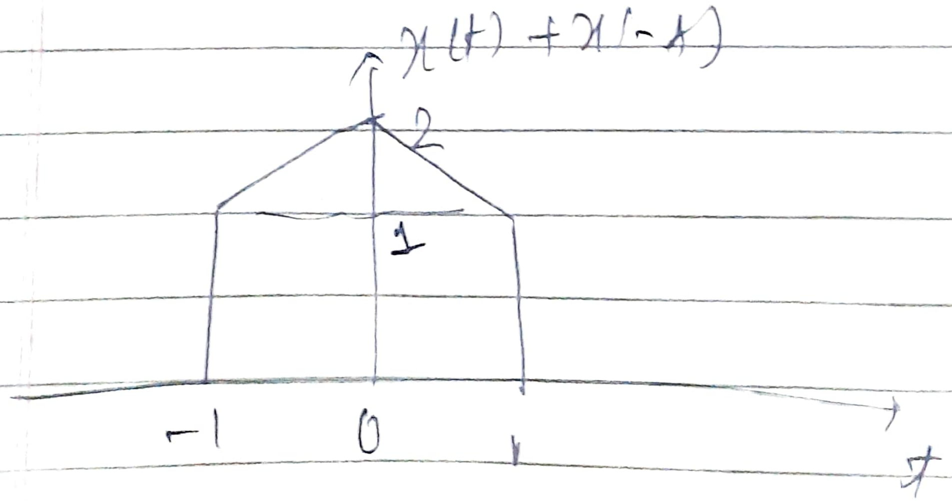
$x(-t) = 1 - t\cos t - t^2\sin t + t^3\sin t\cos t$

$x(t) + x(-t) = 2 + 2t^3\sin t\cos t$

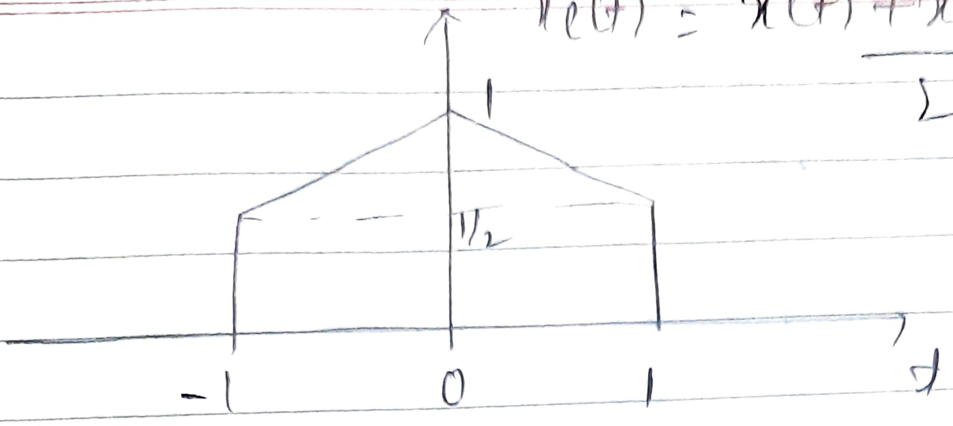
$x_e(t) = 1 + t^3\sin t\cos t$
$x_o(t) = t\cos t + t^2\sin t$

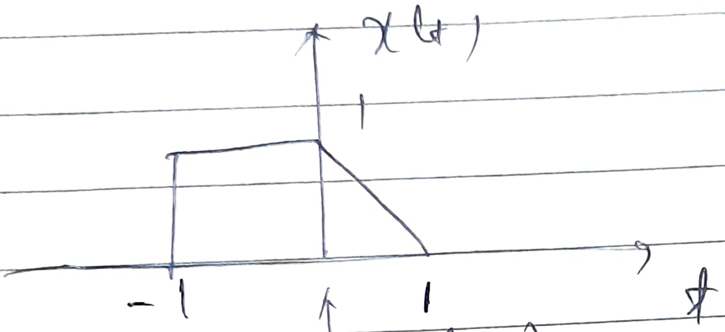


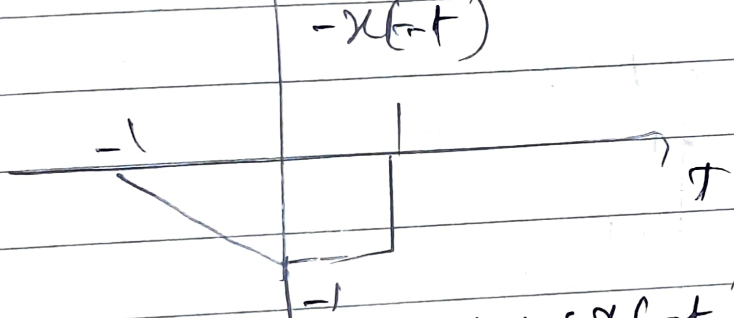


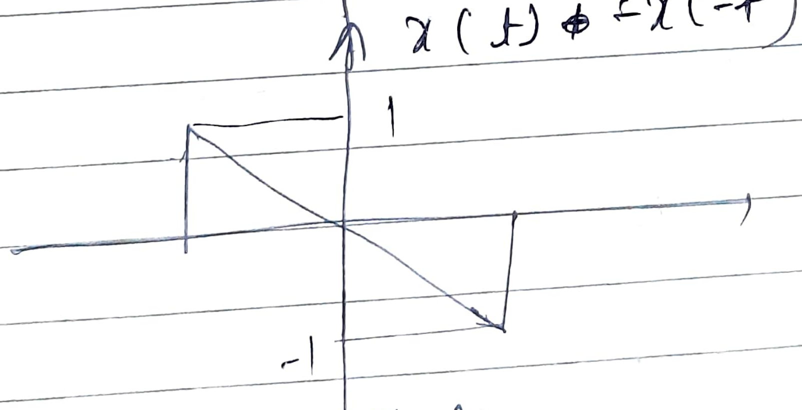


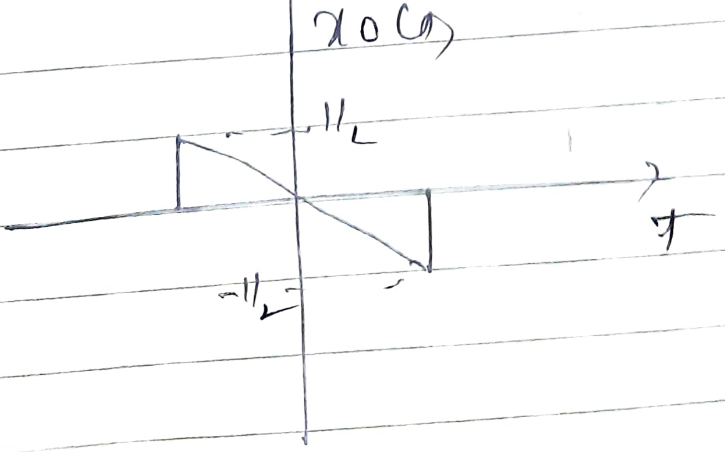
$$x_e(t) = \frac{x(t) + x(-t)}{2}$$











Problems on system propert

VI For each of the system, find wheth the system is linear, shift-invariant, stable, causal, invertible

① $y(n) = \log(x|n|)$

(1) Linear

(i) Let $x(n) = x_1(n)$ & the o/p

$$y_1(n) = \log(x_1|n|)$$

(ii) Let $x(n) = x_2(n)$ & the o/p

$$y_2(n) = \log(x_2|n|)$$

(iii) Let $x(n) = x_3(n) = a_1 x_1(n) + a_2 x_2(n)$
& its o/p

$$y_3(n) = \log(x_3|n|)$$

$$= \log(a_1 x_1|n| + a_2 x_2|n|)$$

$$= \cancel{\log(a_1 x_1|n|) \cdot \log(a_2 x_2|n|)}$$

$$\neq a_1 y_1(n) + a_2 y_2(n)$$

NON linear

(ii) Shift-invariant

(i) Let $x(n) = x_1(n)$ ∴ o/p $y_1(n) = \log x_1(n)$

(ii) Let $y_1(n)$ its o/p be shifted o/p by n_0 units

i.e. $y_1(n-n_0) = \log(x_1(n-n_0))$

(iii) Let $x_2(n) = x_1(n-n_0)$

$$y_2(n) = \log(x_2(n))$$

$$= \log(x_1(n-n_0))$$

$$= y_1(n-n_0)$$

∴ system is Time-invariant

(iii) Stable

Let $|x(n)| \le M_x < \infty$ [BI]

Then its o/p

$y(n) = \log(x|n|)$

$|y(n)| = |\log(x|n|)|$

$\leq \log |x(n)|$

$\leq \log M_x$

$= M_y$ (say) BO.

$< \infty$

The system has bounded o/p as long as the i/p is bounded

∴ the system is Stable

(4) **causal**

$y(n) = \log(x|n|)$

$y(1) = \log(x(1))$

$y(-1) = \log(x|-1|)$

$= \log x(1)$

A system is said to be causal if its present o/p depends on present or past value or combination of present & past values of i/p

hence at $n<0$, the o/p depends upon its future value of the i/p

∴ system is non-causal

(5) invertible :- The system is

invertible since i/p can be recovered from the system's by taking antilog

i.e. $x(n) = e^{(y(n))}$

(2) $y(n) = x(n^3)$

linear

$x(n) = x_1(n)$ & o/p $y_1(n) = x_1(n^3)$
$x(n) = x_2(n)$ & o/p $y_2(n) = x_2(n^3)$
let $x(n) = x_3(n) = a_1 x_1(n) + a_2 x_2(n)$

& its o/p

$y_3(n) = x_3(n^3)$

$= a_1 x_1(n^3) + a_2 x_2(n^3)$

$= a_1 y_1(n) + a_2 y_2(n)$

$\therefore$ system is linear

(2) shift - invariant

(i) Let $x(n) = x_1(n)$ & o/p $y_1(n) = x_1(n^3)$

(ii) Let $y_1(n)$ is shifted by no units

$$y_1(n-no) = x_1((n-no)^3)$$

(iii) Let $x_2(n) = x_1(n-no)$

$$\therefore y_2(n) = x_2(n^3)$$
$$= x_1(n^3-no)$$
$$\neq y_1(n-no)$$

$\therefore$ system is n't shift invariant

(3) **Stable**

Let $|x(n)| \le M_x < \infty$

Then $y(n) = x(n^3)$

$$|y(n)| \le |x(n^3)|$$
$$\le |M_x|$$
$$\le M_y < \infty$$

As long as the i/p $x(n)$ is a bounded i/p its o/p is also bounded

∴ System is stable

(4) **causal**

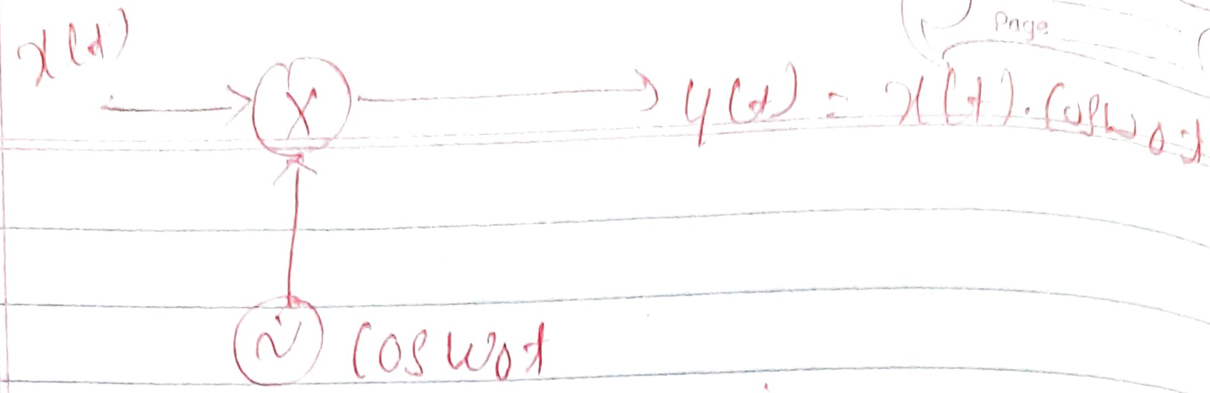
$$y(n) = x(n^3)$$
$$y(1) = x(1)$$
$$y(2) = x(2^3) = x(8)$$

The o/p depands upon its future value of its i/p hence the system is not causal

(5) **Invertible :**

The system is invertible since i/p can be obtained from o/p by properly sampling

③



Determine whether it is
memoryless
causal
linear
Time invariant
stable

$$y(t) = x(t) \cdot \cos w_0 t$$

(i) $y(t)$ depends only on the present value of the i/p $x(t)$, the system under consideration is memoryless

(ii) causal

(iii) linear

(iv) not time invariant

(v) stable $|\cos w_0(t)| \leq 1$,

(5) $x(n) = n\,x(n)$

for $x_1(n) = \delta(n)$, $x_2(n) = u(n)$ for $n = 0$

$y(n) = 0$ — 2 diff i/ps give same o/p ∴ NON invert

(4) determine which of the foll systems are invertible

a. $y(n) = 2x(n)$

invertible since the from o/p $y(n)$ we can recover i/p.

$$x(n) = \frac{1}{2}\,y(n)$$

~~b. $y(n) = x(n) - x(n-1)$~~

~~$x(n)$ & $x(n)+C$ produce the same o/p $y(n)$~~

b) $y(n) = \mathrm{Re}[x(n)]$

$x(n) = 2+j2$

$x(n) = 2-j2$ produces the same o/p

hence the system is not invertible

c) $y(t) = \cos x(t)$ not invertible

since $x(t)$ & $x(t)+2\pi$ given the same o/p

d) $y(t) = x(t+1)$ invertible

$x(t) = y(t-1)$

(8) $T\{x(n)\} = g(n)\,x(n)$ Ep 1.22 (GT) (68)

① stable if $|g(n)| < \infty$

② causal

③ linear

④ not time-invariant

⑤ since o/p $y(n)$ depends only on the nth value of i/p $x(n)$, system under consideration is memoryless

(9) $y(n) = T[x(n)] = \sum_{k=n_0}^{n} x(k)$

(9) as $n \to \infty$, $|y(n)| \to \infty$
$\therefore$ not stable
i.e. unstable

$x(n) = x(n_0) + x(1) + \cdot \cdot - x($

Let $n_0 = 0$, $n = 5$

~~$y(5) = x(0) + x(1) + x(2) + x(3) \cdot \cdot$~~

if $n < n_0$, $y(n)$ depends on the future values of $x(n)$

∴ not causal

c) linear

not time invariant

$n > n_0$, $y(n)$ depends on past value ∴ memory

$(\sin 6t)\, x(t) = y(t)$

① memoryless ∴ present value of $y(t)$ depends on present value of $x(t)$

② not time-invariant

③ linear

④ stable $|\sin 6t| \le 1$

⑤ causal

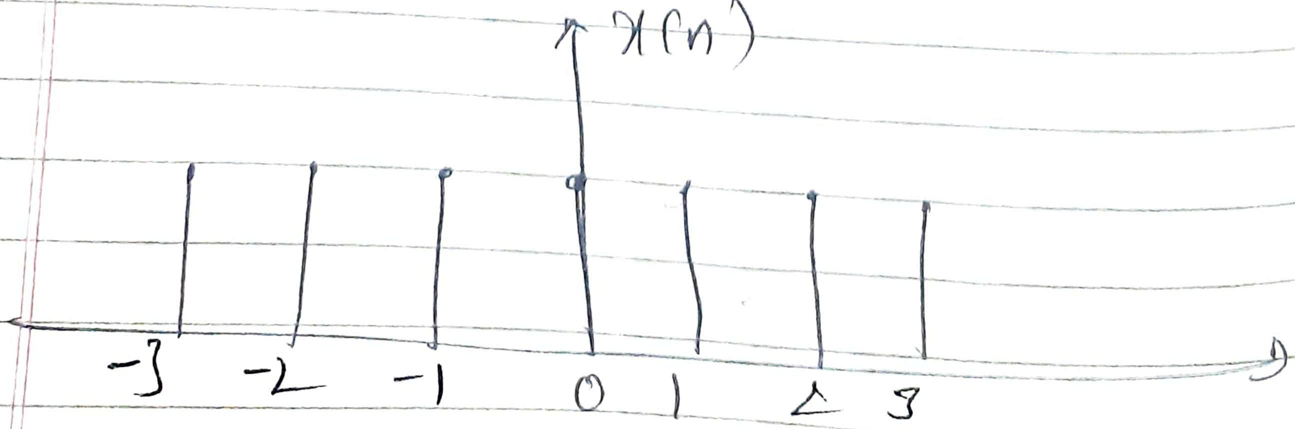
problems on sketching of signals

① sketch each of the foll signals

ⓐ $x_1(n) = \text{rect}\left(\frac{n}{6}\right)$

$\text{rect}\left(\frac{n}{2N}\right)$

$2N = 6$ ∴ $x_1(n)$ has 7 samples



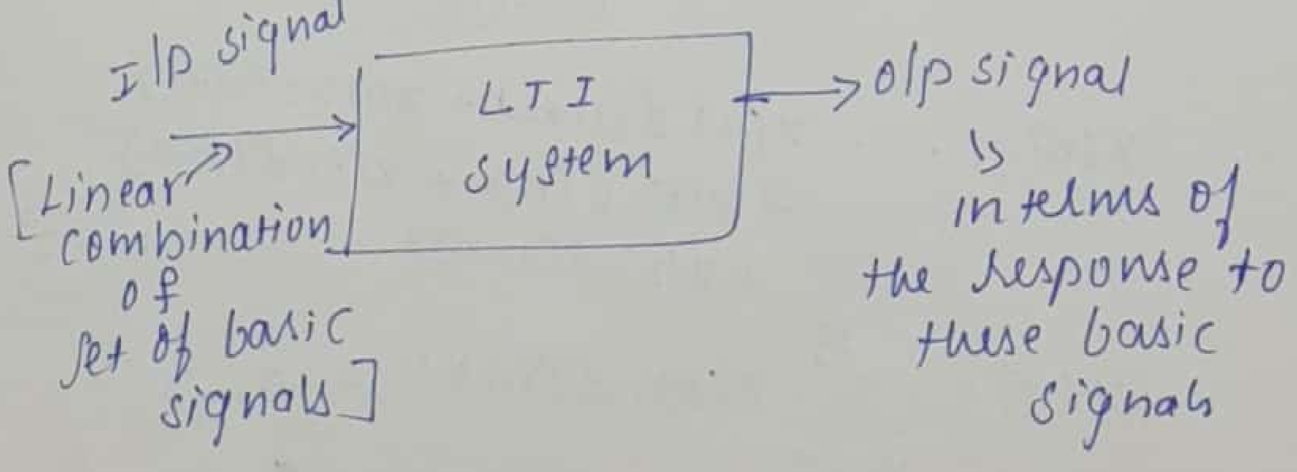
TIME-DOMAIN REPRESENTATION OF LTI SYSTEM

* The system which statisfy linearity and time-invariance properties are called as Linear-Time invariant (LTI) system

* In this module we will study the different time-domain representation of LTI system. i.e, the representation of that relates the output signal to the input signal when both the signals are represented as a function of time.

* There are different methods for representing an LTI system in time domain

* Three of them are
  (i) In terms of Impulse response
  (ii) using differential / difference equation
  (iii) using Block Diagram representation



Impulse Response Representations for LTI systems

* A complete charaterstics of any LTI system can be represented in terms of its response to a unit impulse, known as impulse response of the system

Impulse response of discrete-time LTI systems

Let us consider any discrete time signal x[n] in term of time shifted unit impulses

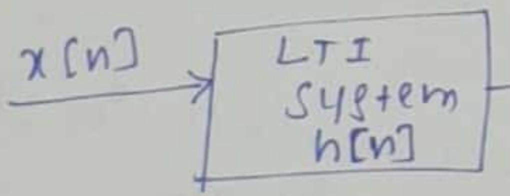


fig ①

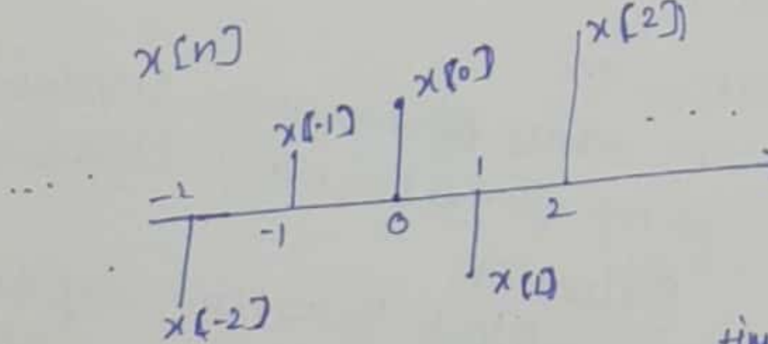


fig ② i/p signal represented interms of time shifted unit impulse

$$y[n] = T\{x[n]\} \quad \text{①}$$

$T\{\cdot\}$ = system operator

$$x[n] = \dots x[-2]\,\delta(n+2) + x(-1)\,\delta(n+1) + x(0)\,\delta(n) + x(1)\,\delta(n-1) + x(2)\,\delta(n-2) + \dots \quad \text{②}$$

$$x[n] = \sum_{k=-\infty}^{\infty} x(k)\cdot\delta(n-k) \longrightarrow \text{③}$$

from eq ①

$y[n] = T\{x[n]\}$

substituting for $x[n]$ from eq ③

$$y[n] = T\left\{\sum_{K=-\infty}^{\infty} x(K).\,\delta(n-k)\right\}$$

$$= \sum_{K=-\infty}^{\infty} x(k).\,T\{\delta(n-k)\} \rightarrow ④$$

(using lineality)

Let $T\{\delta(n)\} = h(n)$ = impulse response of the system

then $T\{\delta(n-k)\} = h(n-k)$

$$\therefore y[n] = \sum_{K=-\infty}^{\infty} x(k).\,h(n-k) \quad — ⑤$$

$$y(n) = x[n] * h[n] \quad — ⑥$$

$\therefore$ convolution sum

from eq ⑤ the o/p of an LTI system is given by a weighted superposition of time-shifted impulse response & this is known as convolution sum

sum of the following signals

Properties of Convolution Sum

① Commutative property

$$x[n] * h[n] = h[n] \times x[n]$$

Proof:

$$x(n] * h[n] \triangleq \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k)$$

Replace $(n-k) = m$

$$\therefore k = n-m$$

$$x(n] * h[n] = \sum_{m=+\infty}^{-\infty} x(n-m)\, h(m)$$

$$= \sum_{m=-\infty}^{+\infty} h(m) \cdot x(n-m)$$

replace m by k

$$= \sum_{k=-\infty}^{\infty} h(k) \cdot x(n-k)$$

$$x[n] * h[n] = h[n] * x[n]$$

Associative Property

$$\underbrace{\left[x[n] * h_1[n]\right]}_{f_1(n)} * h_2[n] = x(n) * \underbrace{\left[h_1(n) * h_2(n)\right]}_{f_2(n)}$$

Proof

Let x $f_1[n] = x(n) * h_1(n)$

$f_2[n] = h_1[n] * h_2[n]$

$$f_1[n] \triangleq \sum_{K=-\infty}^{\infty} x(K) \cdot h_1(n-K) \quad \text{—①}$$

L.H.S

$$\left[x[n] * h_1[n]\right] * h_2[n] = f_1(n) * h_2(n)$$

$$\triangleq \sum_{m=-\infty}^{\infty} f_1(m)\, h_2(n-m) \quad \text{—②}$$

using eq ① in eq ②

$$= \sum_{m=-\infty}^{\infty} \left\{ \sum_{K=-\infty}^{\infty} x(K) \cdot h_1(m-k) \right\} h_2(n-m) \quad \text{—③}$$

Let $r = m-k$ in eq ③ & interchanging the order of summation

$$= \sum_{K=-\infty}^{\infty} x(k) \sum_{r=-\infty}^{\infty} h_1(r) \cdot h_2(n-[r+k])$$

$$h_2(n-r-k) \quad \text{—④}$$

of the following signals

wkt $f_2[n] = h_1[n] * h_2[n]$

$$\triangleq \sum_{r=-\infty}^{\infty} h_1(r)\cdot h_2(n-r)$$

$$\therefore f_2[n-k) = \sum_{r=-\infty}^{\infty} h_1(r)\, h_2(n-k-r) \rightarrow ⑤$$

using eq ⑤ in ④

$$[x(n] * h_1[n]] * h_2[n] = \sum_{k=-\infty}^{\infty} x(k)\cdot f_2(n-k)$$

$$= x[n] * f_2[n]$$

$$= x[n] * [h_1[n] * h_2[n]]$$

③ Distributive property

$$x[n] * [h_1[n] + h_2[n]] = [x[n] * h_1[n]] + [x[n] * h_2[n]]$$

$$x[n] * \{h_1[n] + h_2(n)\} \triangleq \sum_{k=-\infty}^{\infty} x(k)\cdot[h_1(n-k) + h_2(n-k)]$$

$$= \sum_{k=-\infty}^{\infty} x(k)\cdot h_1(n-k) + \sum_{k=\infty}^{\infty} x(k)\cdot h_2(n-k)$$

$$= [x(n) * h_1(n)] + [x[n] + h_2[n]]$$

④ $x[n] * \delta(n-k) = x[n-k)$

$$\boxed{\delta(n-k) * \delta(n-m) = \delta[n-(k+m)]}$$

4

STEPS TO FIND CONVOLUTION

1> Plot the Given signals & define them

2> Define convolution sum for the given ~~sig~~ signals i.e $y(n) = x(n) * h(n)$ (or) $h(n) * x(n)$

$$= \sum_{K=-\infty}^{\infty} x(k).h(n-k) \text{ or } \sum_{K=-\infty}^{\infty} h(K).x(n-k)$$

3> Define & Sketch $x(K)$ & $h(n-K)$

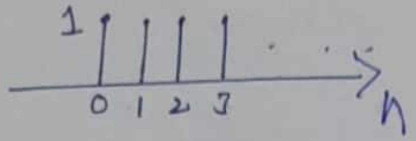
4> compute the convolution for diffelent cases by (i) fixing the 1st signal & (ii) moving the second signal thlo' out the time.

Find the convolution sum of the following signals

① $x(n) = u(n)$ & $h(n) = u(n)$

I

I> $x(n) = h(n) = u(n)$ $\quad x(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$



II> WKT $y(n) = x(n) * h(n)$

$$\triangleq \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k)$$

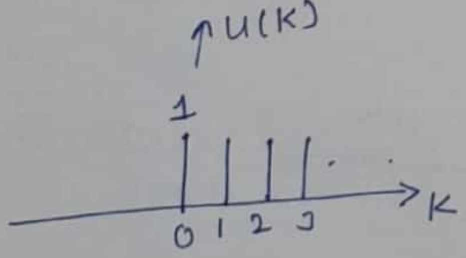
$$= \sum_{k=-\infty}^{\infty} u(k) \cdot u(n-k)$$

$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$

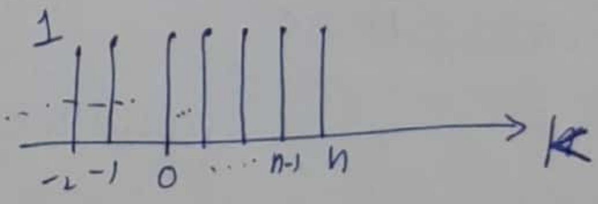
III> define $u(k)$ & $u(n-k)$

$u(k) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$

$u(n-k) = \begin{cases} 1, & n-k \geq 0 \text{ (or) } -k \geq -n \text{ (or) } k \leq n \\ 0, & k > n \end{cases}$



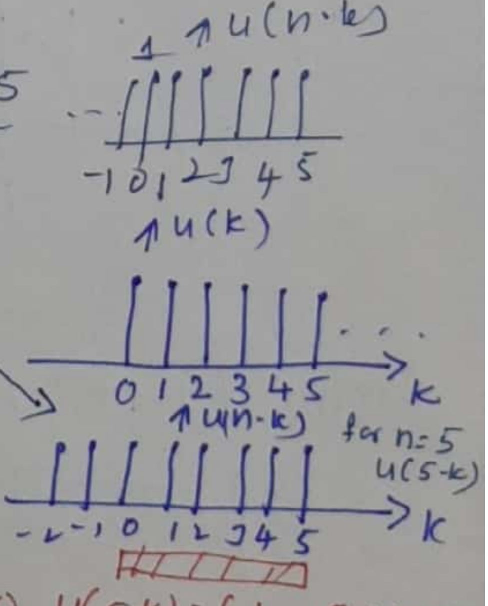
u(n-k)



eg! assume $n=5$ then

$u(n-k) = \begin{cases} 1, & k \leq 5 \\ 0, & k > 5 \end{cases}$

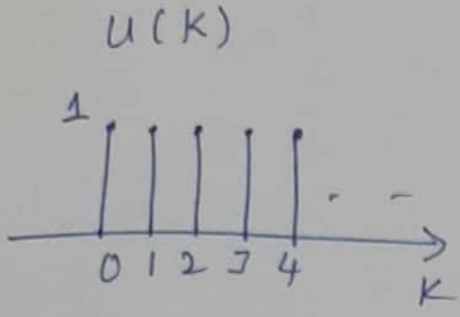
$u(k) \cdot u(n-k) = \begin{cases} 1, & 0 \leq k \leq n \\ 0, & \text{elsewhere} \end{cases}$

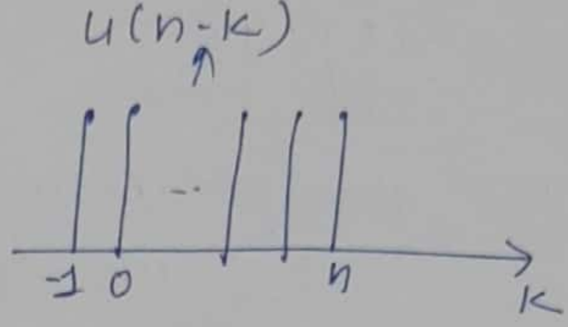


$u(k) \cdot u(5-k) = \begin{cases} 1, & 0 \leq k \leq 5 \\ 0, & \text{el} \end{cases}$

WKT $y(n) = \sum_{k=-\infty}^{\infty} x(k).h(n-k)$

$= \sum_{k=-\infty}^{\infty} u(k).u(n-k)$

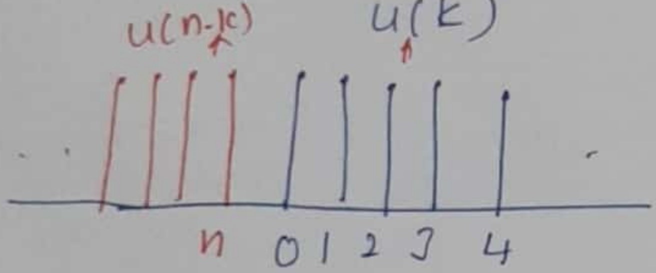




IV

Case

(i) $\boxed{n<0}$



NO ~~overal~~ overlapping of u(k) & u(n-k)

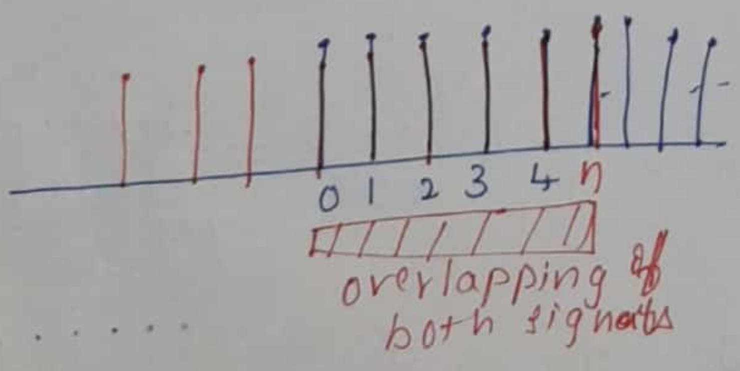
∴ $\boxed{y(n)=0}$

using blue ink pen → u(k) is sketched

—" red " → u(n-k) —

$\boxed{\text{case(ii)} \quad n>0}$

u(k).u(n-k)



∴ $y(n) = \sum_{k=0}^{n} u(k).u(k-1)$

$= \sum_{k=0}^{n} 1.1 = \sum_{k=0}^{n} 1$

$\boxed{y(n) = n+1}$

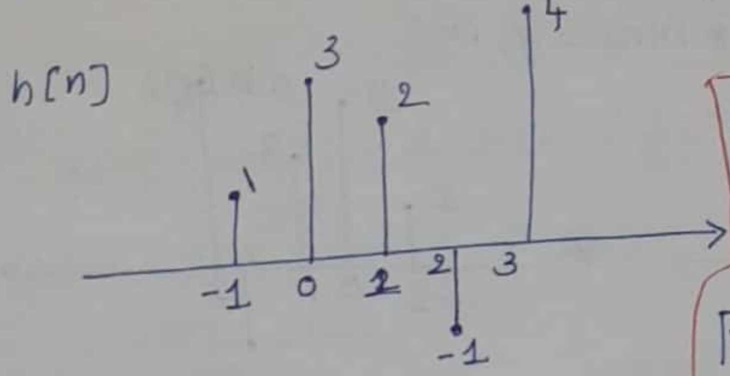
$$\boxed{y(n) = \begin{cases} n+1 ; & n>0 \\ 0, & n<0 \end{cases}}$$

eg: $\sum_{k=0}^{n} 1$ assume n=5

Then $\sum_{k=0}^{5} 1 = \underset{k=0}{\underset{\uparrow}{1}} + \underset{1}{\underset{\uparrow}{1}} + \underset{2}{\underset{\uparrow}{1}} + \underset{3}{\underset{\uparrow}{1}} + \underset{4}{\underset{\uparrow}{1}} + \underset{5}{\underset{\uparrow}{1}} = 6 = (5+1)$

2

② $x[n] = 2\delta[n] - \delta[n-1]$ using graphical method.



$h[n] = [1, \underset{\uparrow}{3}, 2, -1, 4]$

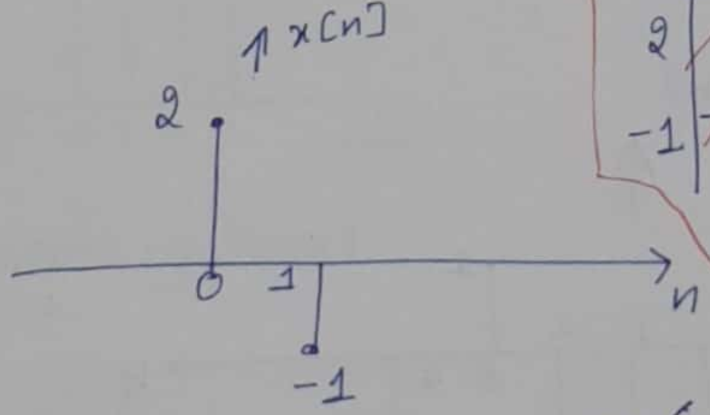
$x[n] = [\underset{\uparrow}{2}, -1]$

Verification

	1	3	2	-1	4
2	2	6	4	-2	8
-1	-1	-3	-2	1	-4

$y(n) = \{2, \underset{\uparrow}{5}, 1, -4, 9, -4\}$

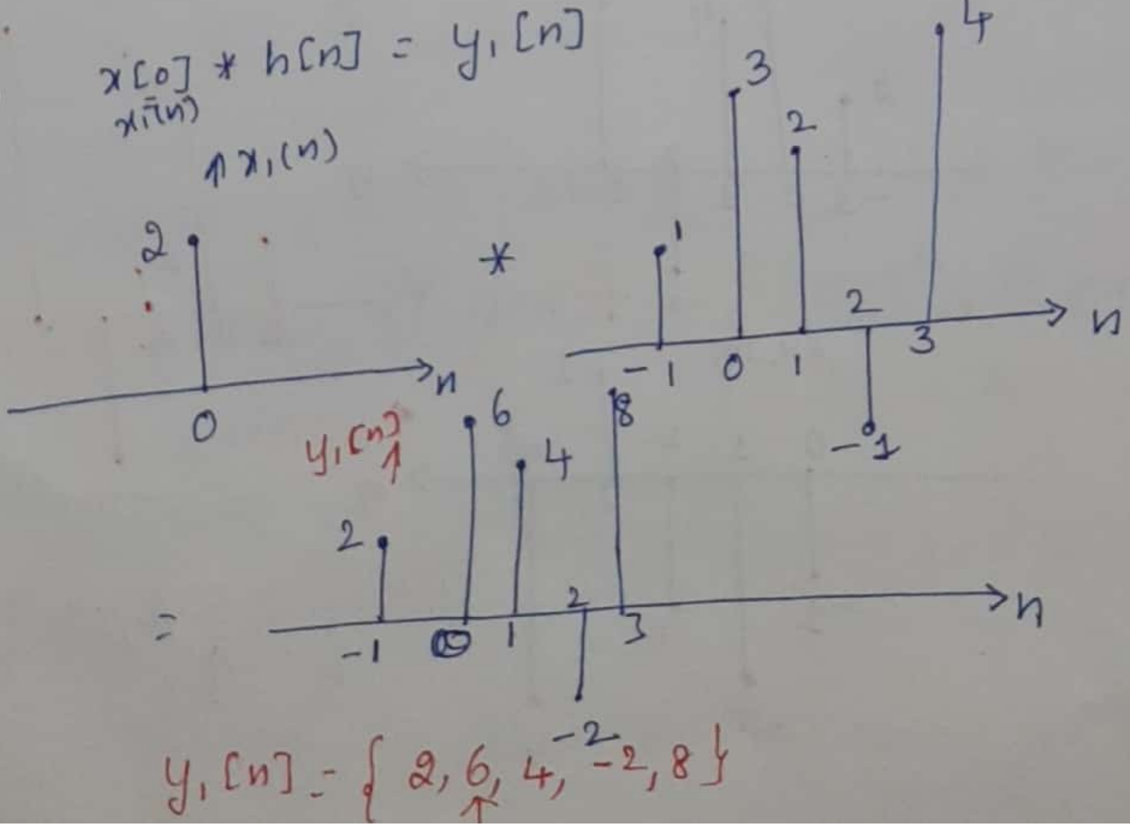
I sketch x[n]



WKT $\delta(n-\underline{k}) * \delta(n-\underline{m}) = \delta(n - [k+\underline{m}])$

II convolute each sample / impulse of x[n] with h[n] & find o/p

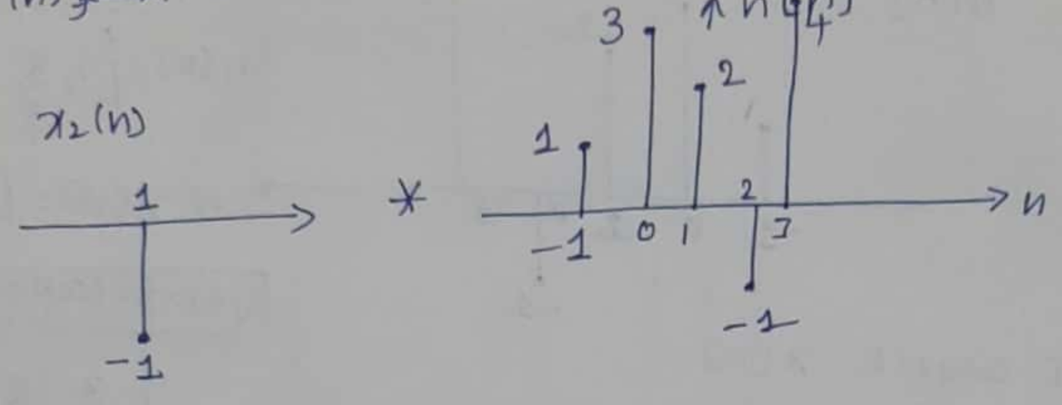
(i) $x[0] * h[n] = y_1[n]$
$x_1[n]$



$y_1[n] = \{2, \underset{\uparrow}{6}, 4, \overset{-2}{-2}, 8\}$

(ii) x_2 ~~$(n) = x_1(1) * h_2$~~

$(x_2(n)$ ~~$*$~~ $x_1(1) * h(n) = y_1(n)$



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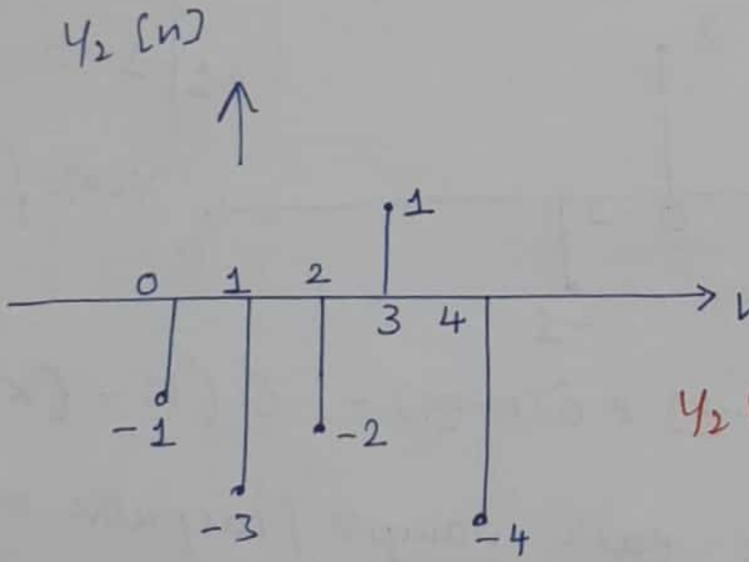
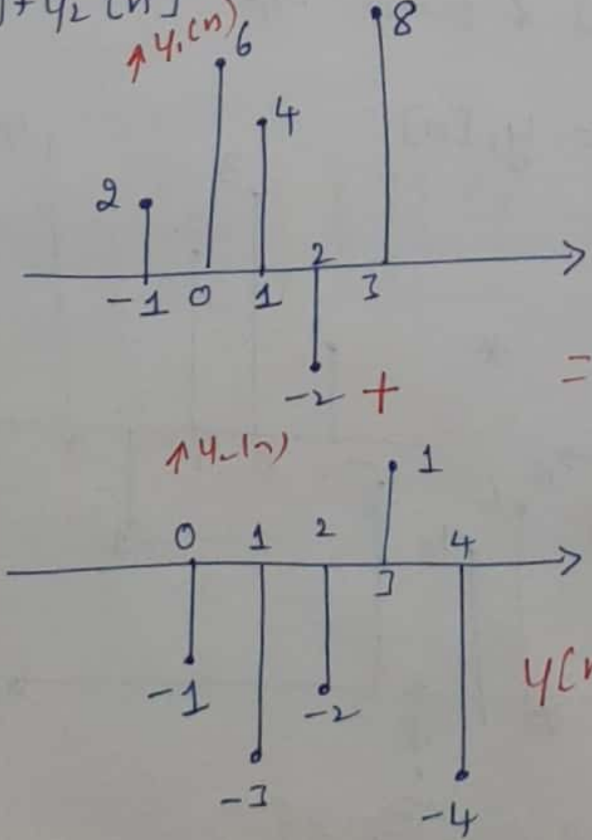


$y_2[n] = \{-1, -3, -2, 1, -4\}$
$\uparrow$

$y[n] = y_1[n] + y_2[n]$



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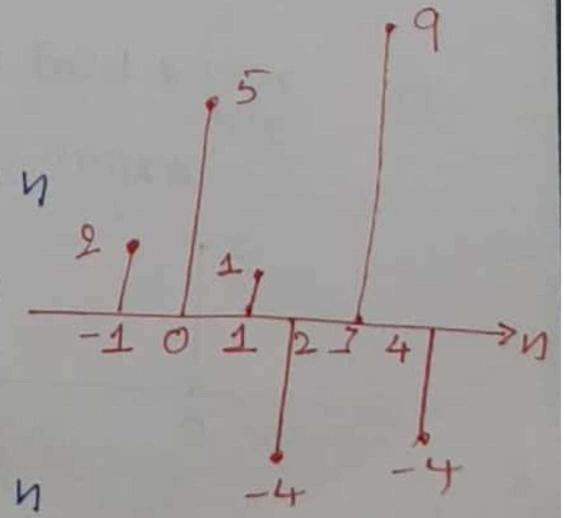


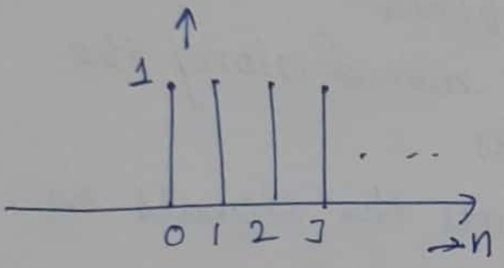
$y[n] = \{2, 5, 1, -4, 9, -4]$
$\uparrow$

③ The impulse response for LTI system is given by³ $h(n) = \left(\frac{1}{2}\right)^n u(n)$. find the step response.

$x[n] = u[n]$ & $h[n] = \left(\frac{1}{2}\right)^n u[n]$

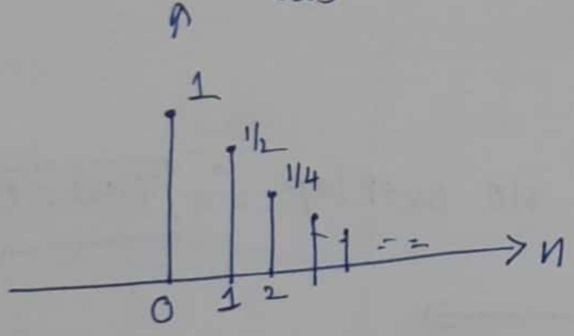
I plot the given signals & define them

$x[n] = u(n)$



$$x[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$h(n) = \left(\frac{1}{2}\right)^n u(n)$



$$h(n) = \begin{cases} \left(\frac{1}{2}\right)^n & n \geq 0 \\ 0, & n < 0 \end{cases}$$

II $y(n) = x(n) * h(n) = h(n) * x(n)$

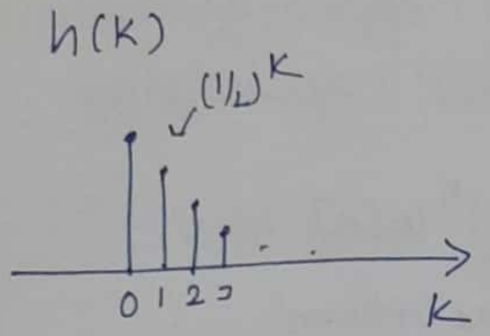
$$\triangleq \sum_{k=-\infty}^{\infty} h(k) \cdot x(n-k)$$

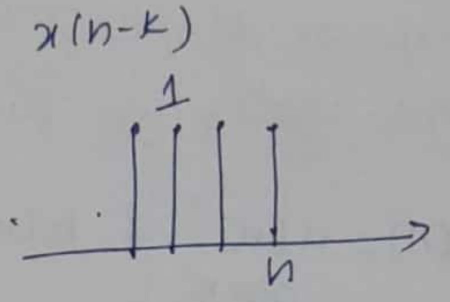
III Again plot & redefine $h(k)$ & $x(n-k)$

$$h(k) = (1/2)^k u(k) = \begin{cases} (1/2)^k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

$$x(n-k) = \begin{cases} 1, & \overset{\frown}{n-k \geq 0} \text{ (or) } -k \geq -n \text{ (or) } \boxed{k \leq n} \\ 0, & k > n \end{cases}$$

Plot $x(n-k)$ (i) reflect the signal
& (ii) Then shift the reflected signal by 'n' units ②

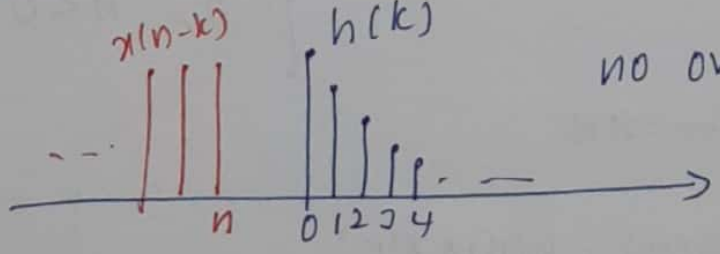




IV start convoluting both signal.

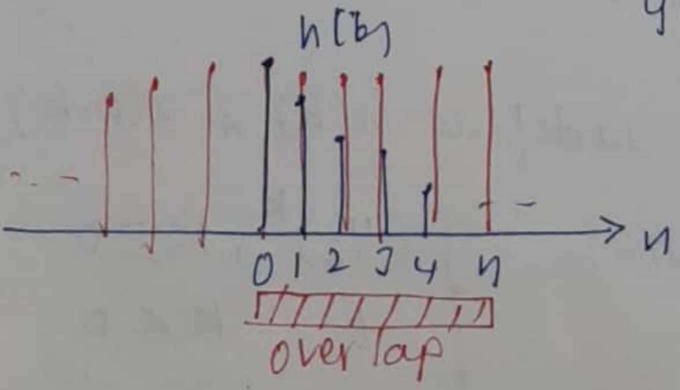
(i) first signal is fixed
(ii) second signal is moved along the entire k-axis
* multiply & evaluate the signals on the overlapped range.

case(i) $n<0$



no overlap ∴ $\boxed{y(n)=0}$

case (ii) $n>0$



$$y(n)=\sum_{K=0}^{n} h(K)\cdot x(n-K)$$

$$=\sum_{K=0}^{n} (1/2)^{K}\cdot 1$$

$$=\sum_{K=0}^{n} (1/2)^{K}$$

WKT = $\cancel{\sum_{K=0}^{N} a}$ $\boxed{\sum_{K=0}^{N} a^{K} = \frac{1-a^{N+1}}{1-a}\ ;\ a\neq 1}$

$$\therefore \quad y(n) = \sum_{k=0}^{n} (1/2)^k$$

$$\left[\frac{1 - a^{n+1}}{1-a} \quad \text{here } a = 1/2 \right]$$

$$= \frac{1 - (1/2)^{n+1}}{1 - 1/2} = \frac{1 - (1/2)^{n+1}}{\left(\frac{2-1}{2}\right)}$$

$$= 2\left[1 - (1/2)^{n+1}\right] u(n)$$

$$\therefore \quad \boxed{y(n) = \begin{cases} 2\left[1 - (1/2)^{n+1}\right] & n \geq 0 \\ 0, & n < 0 \end{cases}}$$

(or)

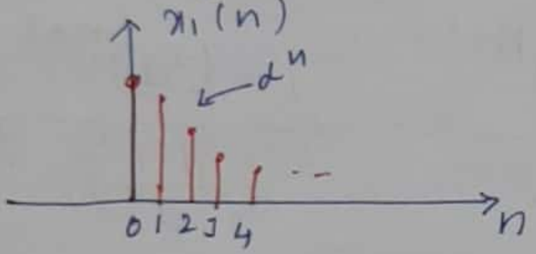
$$\boxed{y(n) = 2\left[1 - (1/2)^{n+1}\right] u(n)}$$

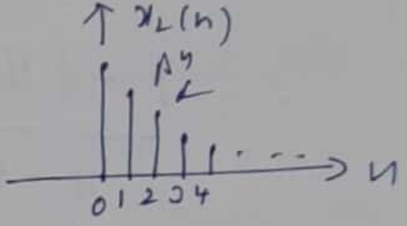
4) $x_1(n) = \alpha^n u(n)$ $\quad x_2(n) = \beta^n u(n)$

$\boxed{X}$

I plot the given signals & define them

$$x_1(n) = \begin{cases} \alpha^n, & n \geq 0 \\ 0, & n < 0 \end{cases} \qquad x_2(n) = \begin{cases} \beta^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$





II $y(n) = x_1(n) * x_2(n)$

$$\triangleq \sum_{k=-\infty}^{\infty} x_1(k) \cdot x_2(n-k)$$

$$= \sum_{k=-\infty}^{\infty} \alpha^k u(k) \cdot \underbrace{\beta^{(n-k)}}_{\frac{\beta^n}{\beta^k}} u(n-k)$$

$$= \sum_{k=-\infty}^{\infty} \alpha^k u(k) \cdot \frac{\beta^n}{\beta^k} u(n-k)$$

$$= \beta^n \sum_{k=-\infty}^{\infty} \left(\frac{\alpha}{\beta}\right)^k u(k) \cdot u(n-k)$$

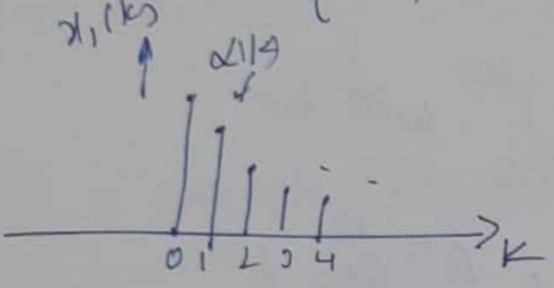
WKT $u(k) \cdot u(n-k) = \begin{cases} 1, & 0 \leq k \leq n \\ 0, & \text{elsewhere} \end{cases}$

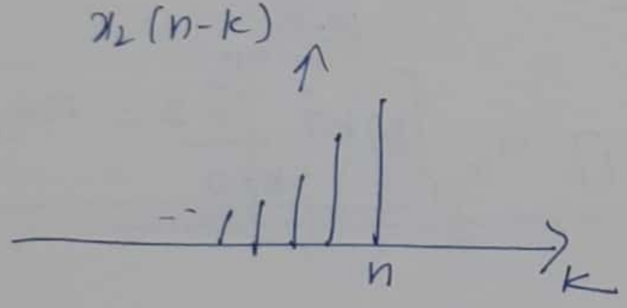
$$y(n) = \beta^n \sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^k \text{———} ①$$

⑤

III Again plot & redefine $x_1(k)$ & $x_2(n-k)$

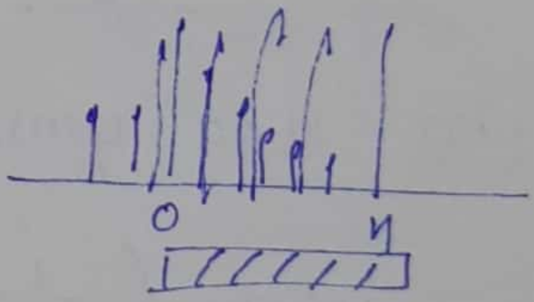
$$x_1(k) = \begin{cases} \alpha^k & k \geq 0 \\ 0, & k < 0 \end{cases} \qquad x_2(n-k) = \begin{cases} \beta^{(n-k)}, & n-k \geq 0 \text{ or } -k \geq -n \text{ or } \boxed{k \leq n} \\ 0, & k > n \end{cases}$$





IV

case (i) $\boxed{n \geq 0 \;\&\; \alpha \neq \beta}$



use eq ①

$$y(n) = \beta^n \sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^k$$

WKT $$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$$

$$\therefore\; y(n) = \beta^n \left[\frac{1-(\alpha/\beta)^{n+1}}{1-(\alpha/\beta)} \right]$$

$$= \beta^n \left[\frac{1-(\alpha/\beta)^{n+1}}{1-\alpha/\beta} \right] u(n) \quad (\text{or})$$

can be simplified further

$$= \frac{1}{\beta-\alpha}\left[\beta^{n+1} - \alpha^{n+1}\right]$$

(ii) $n \geq 0$ & $\alpha = \beta$

$$y(n) = \beta^{n} \sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^{k} \text{ will be}$$

$$= \beta^{n} \sum_{k=0}^{n} 1^{k} = \beta^{n} \sum_{k=0}^{n} 1$$

$$= \beta^{n}[n+1] \qquad \boxed{\text{WKT } \sum_{k=0}^{n} 1 = n+1}$$

(iii) $n < 0$ $y(n) = 0$

$$y[n] = \begin{cases} \frac{1}{\beta-\alpha}\left[\beta^{n+1} - \alpha^{n+1}\right] & ; n \geq 0 \text{ \& } \alpha \neq \beta \\ \beta^{n}[n+1] & ; n \geq 0 \text{ \& } \alpha = \beta \\ 0, & ; n < 0 \end{cases}$$

Solving/simplifying

$$y(n) = \beta^{n}\left[\frac{1-\left(\frac{\alpha}{\beta}\right)^{n+1}}{1-(\alpha/\beta)}\right]$$

$$= \beta^{n}\left[\frac{\beta^{n+1} - \alpha^{n+1}/\beta^{n+1}}{\beta-\alpha/\beta}\right]$$

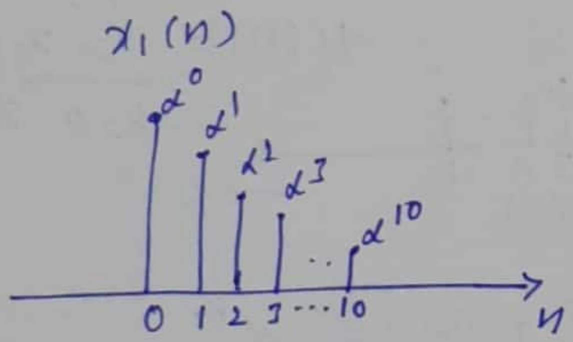
$$= \frac{\beta^{n}\cdot\beta\left[\beta^{n+1} - \alpha^{n+1}\right]}{\beta^{n+1}[\beta-\alpha]} = \frac{\cancel{\beta^{n+1}}}{\cancel{\beta^{n+1}}} \cdot \frac{\beta^{n+1} - \alpha^{n+1}}{\beta-\alpha}$$

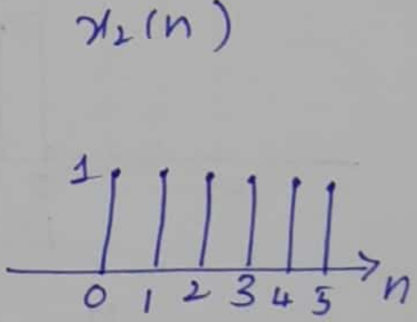
$$= \frac{1}{\beta-\alpha}\left[\beta^{n+1} - \alpha^{n+1}\right]$$

⑤ $x_1(n) = \begin{cases} \alpha^n, & 0 \leq n \leq 10 \\ 0, & \text{elsewhere} \end{cases}$

$x_2(n) = \begin{cases} 1, & 0 \leq n \leq 5 \\ 0, & \text{elsewhere} \end{cases}$

I sketch the signals



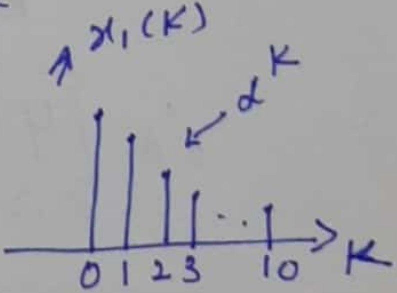


II $y(n) = x_1(n) * x_2(n)$

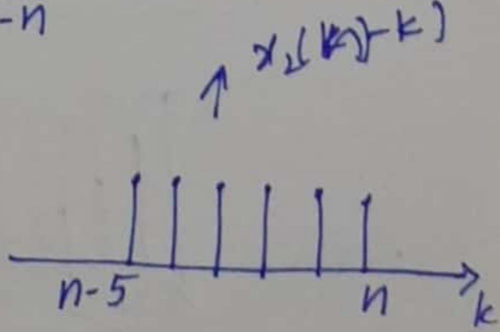
$$\triangleq \sum_{K=-\infty}^{\infty} x_1(K) \cdot x_2(n-K)$$

III plot & redefine signals WRT "K"

$x_1(K) = \begin{cases} \alpha^K, & 0 \leq K \leq 10 \\ 0, & \text{elsewhere} \end{cases}$

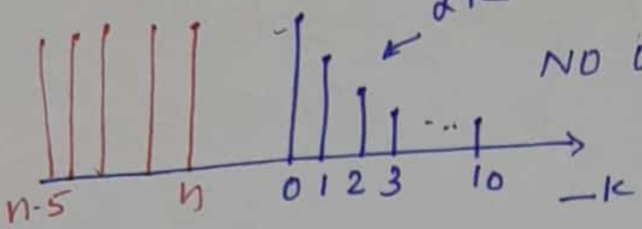


$x_2(n-K) = \begin{cases} 1, & 0 \leq n-K \leq 5 \\ & \text{(or)} \\ & -n \leq -K \leq 5-n \\ & \text{(or)} \\ & \boxed{n-5 \geq K \geq n} \\ 0, & \text{elsewhere} \end{cases}$



IV

(i) $\boxed{n<0}$

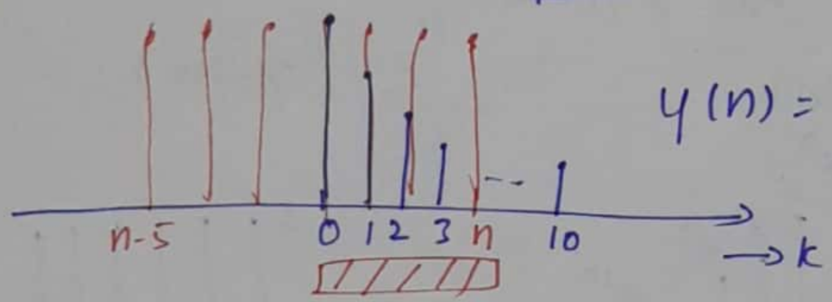


NO overlap

$\boxed{\therefore y(n)=0}$

(ii) $\boxed{n>0}$ & $\boxed{n-5<0} \Rightarrow \boxed{n<5}$

$\boxed{0<n<5}$



$$y(n) = \sum_{k=0}^{n} x_1(k) \cdot x_2(n-k)$$

$$y(n) = \sum_{k=0}^{n} \alpha^{k} \cdot 1 = \sum_{k=0}^{n} \alpha^{k}$$

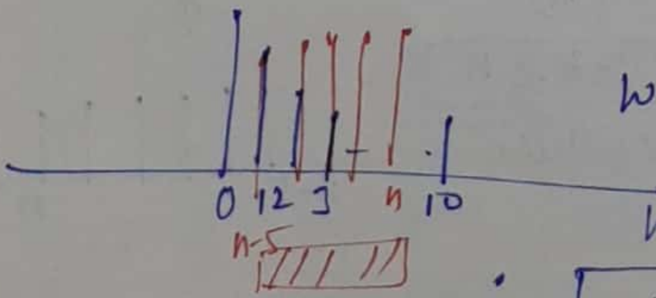
WKT $\sum_{k=0}^{N} a^{k} = \frac{1-a^{N+1}}{1-a}$; $a \neq 1$

$$\therefore \boxed{y(n) = \frac{1-\alpha^{n+1}}{1-\alpha}}$$

(iii) $\boxed{n-5>0}$ & $\boxed{n<10}$

$\boxed{5<n<10}$

$$y(n) = \sum_{k=n-5}^{n} \alpha^{n}$$



WKT $\sum_{n=N1}^{N_2} a^{n} = \frac{a^{N1} - a^{N_2+1}}{1-a}$

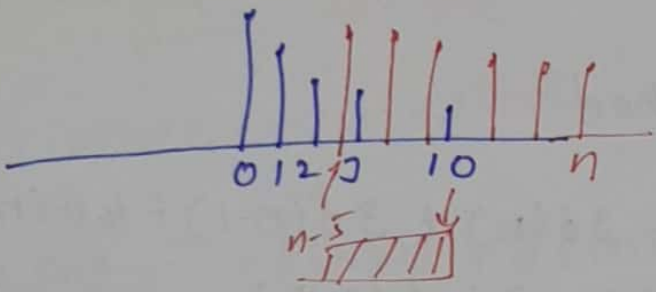
here $N_1 = n-5$ & $N_2 = n$ $a = \alpha$

$$\therefore \boxed{y(n) = \frac{\alpha^{n-5} - \alpha^{n+1}}{1-\alpha}}$$

⑦

(iv) $\underline{n > 10}$ & $n-5 < 10$: $n < 15$

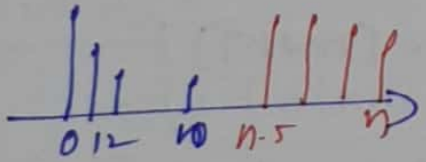
$$\boxed{10 < n < 15}$$



$$y(n) = \sum_{k=n-5}^{10} \alpha^{k} = \frac{\alpha^{n-5} - \alpha^{11}}{1-\alpha}$$

(v) $n > 15$

$$\boxed{y(n) = 0}$$



$$y(n) = \begin{cases} 0, & ; n < 0 \\ \dfrac{1-\alpha^{n+1}}{1-\alpha} & ; 0 < n < 5 \\ \dfrac{\alpha^{n-5} - \alpha^{n+1}}{1-\alpha} & ; 5 < n < 10 \\ \dfrac{\alpha^{n-5} - \alpha^{11}}{1-\alpha} & ; 10 < n < 15 \\ 0 & ; n > 15 \end{cases}$$

⑥ $x_1[n] = \{1, \underset{\uparrow}{2}, 3, 4\}$

$x_2[n] = \{1, 1, 1\}$

find the convolution sum of the $x_1(n)$ & $x_2(n)$ using the foll methods

$\boxed{(1)\ \text{Analytical method}}$ $y[n] = \{1, \underset{\uparrow}{3}, 6, 9, 7, 4\}$

$x_1[n] = 1\delta(n+1) + 2\delta(n) + 3\delta(n-1) + 4\delta(n-2)$

$x_2[n] = 1\delta(n) + 1\delta(n-1) + 1\delta(n-2)$

$y(n) = x_1(n) * x_2(n) = x_2(n) * x_1(n)$

$[\underset{1}{\delta(n)} + \underset{2}{\delta(n-1)} + \underset{3}{\delta(n-2)}] * [\delta(n+1) + 2\delta(n) + 3\delta(n-1) + 4\delta(n-2)]$

$$\boxed{\delta(n-K) * \delta(n-m) = \delta(n-[K+m])}$$

1 $\delta(n) * [\delta(n+1) + 2\delta(n) + 3\delta(n-1) + 4\delta(n-2)]$

$y_1(n) = \delta(n+1) + 2\delta(n) + 3\delta(n-1) + 4\delta(n-2)$

2 $\delta(n-1) * [\delta(n+1) + 2\delta(n) + 3\delta(n-1) + 4\delta(n-2)]$

$y_2(n) = \delta(n) + 2\delta(n-1) + 3\delta(n-2) + 4\delta(n-3)$

3. $\delta(n-2) * [\delta(n+1) + 2\delta(n) + 3\delta(n-1) + 4\delta(n-2)]$

$y_3(n) = \delta(n-1) + 2\delta(n-2) + 3\delta(n-3) + 4\delta(n-4)$

$y(n) = y_1(n) + y_2(n) + y_3(n)$

$= \delta(n+1) + \underline{2}\delta(n) + 3\delta(n-1) + 4\delta(n-2)$

$+ \delta(n) + 2\delta(n-1) + 3\delta(n-2) + 4\delta(n-3)$

$+ \delta(n-1) + 2\delta(n-2) + 3\delta(n-3) + 4\delta(n-4)$

$y(n) = 1\delta(n+1) + 3\delta(n) + 6\delta(n-1) + 9\delta(n-2) +$

$+ 7\delta(n-3) + 4\delta(n-4)$

1. $\sum_{K=0}^{n} 1 = n+1$

2. $\sum_{K=0}^{n} a^K = \frac{1-a^{n+1}}{1-a} \quad |a| \neq 1$

3. $\sum_{K=0}^{\infty} a^K = \frac{1}{1-a} \quad |a| \neq 1$

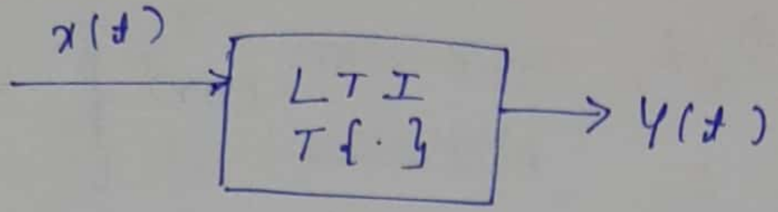
4. $\sum_{K=n}^{\infty} a^K = \frac{a^K}{1-a}$

5. $\sum_{K=0}^{\infty} n \cdot a^n = \frac{a}{(1-a)^2}$

6. $\sum_{K=a}^{b} 1 = b-a+1$

7. $\sum_{K=N_1}^{N_2} a^K = \frac{a^{N_1} - a^{N_2+1}}{1-a}$

CONVOLUTION INTEGRAL



$$y(t) = x(t) * h(t)$$

$$\triangleq \int_{\tau=-\infty}^{\infty} x(\tau)\cdot h(t-\tau)\, d\tau$$

Properties

① commutative :- $x(t) * h(t) = h(t) * x(t)$

② Associative :- $x(t) * [h_1(t) * h_2(t)]$
$= [x(t) * h_1(t)] * h_2(t)$

③ Distributive :- $x(t) * [h_1(t) + h_2(t)]$
$= [x(t) * h_1(t)] +$
$[x(t) * h_2(t)]$

Note :- To sketch $h(t-\tau)$
Perfor $h(-\tau)$ & then
shifting by 't' units

① find the convolution of the foll signals

(i) $h(t) = \begin{cases} \alpha^t, & 0 \leq t < 10 \\ 0, & \text{elsewhere} \end{cases}$ (or) $\{$

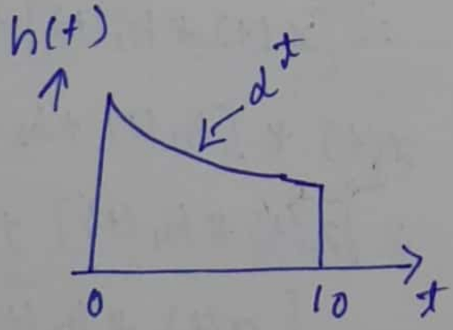
$x(t) = \begin{cases} 1, & 0 < t < 5 \\ 0, & \text{elsewhere} \end{cases}$

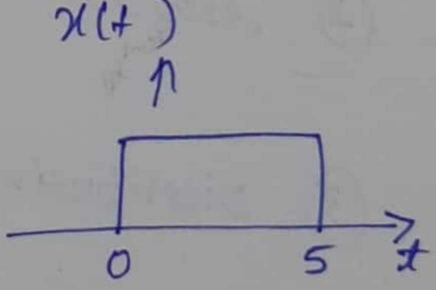
(or)

$h(t) = \alpha^t [u(t) - u(t-10)]$

$x(t) = [u(t) - u(t-5)]$

I sketch the given signal





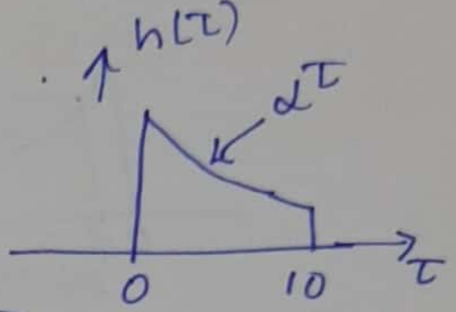
II $y(t) = x(t) * h(t) = h(t) * x(t)$

$$\triangleq \int_{\tau=-\infty}^{\infty} h(\tau) . \, x(t-\tau) \, d\tau$$

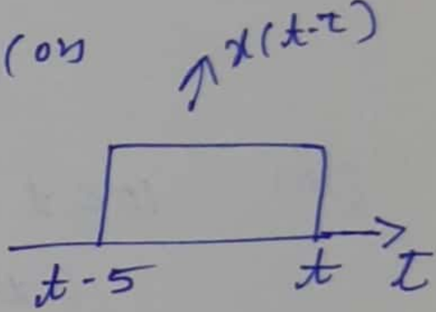
Ⅲ sketch & define $h(\tau)$ & $x(t-\tau)$

$t \to \tau$

$$h(\tau) = \begin{cases} \alpha^{\tau} & 0 < \tau < 10 \\ 0, & \text{elsewhere} \end{cases}$$

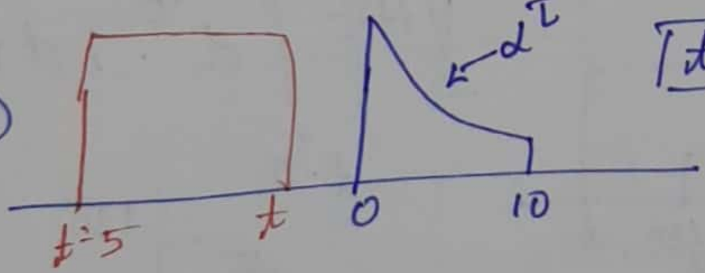


$$x(t-\tau) = \begin{cases} 1, & 0 < t - \tau < 5 \\ & \text{(or)} \\ & -t < -\tau < 5 - t \text{ (or)} \\ & \boxed{t-5 > \tau > t} \\ 0, & \text{elsewhere} \end{cases}$$



Ⅳ

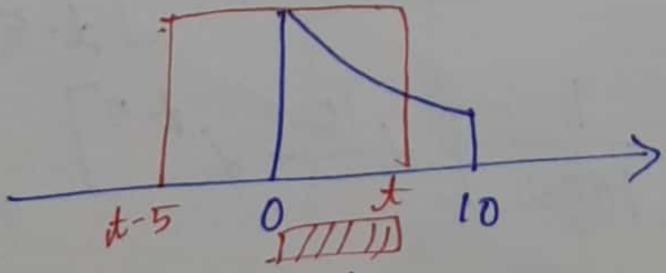
(i)



$\boxed{t < 0}$

$y(t) = 0$

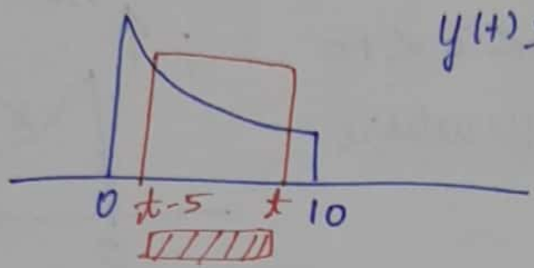
(ii) $\boxed{0 < t < 5}$ $t > 0$ $t - 5 < 0 : t < 5$



$$y(t) = \int_0^t \alpha^{\tau} . 1 \, d\tau = \int_0^t \alpha^{\tau} = d\tau$$

$$= \left. \frac{\alpha^{\tau}}{\log \alpha} \right]_0^t = \frac{1}{\log \alpha} \left[\alpha^t - 1 \right]$$

(ii) $5 < t < 10$ $\left(\begin{matrix} t<10 \\ t-5>0 \end{matrix}\right)$



$$y(t) = \int_{t-5}^{t} \alpha^{\tau} \cdot 1 d\tau$$

$$= \left.\frac{\alpha^{\tau}}{\log\alpha}\right]_{t-5}^{t}$$

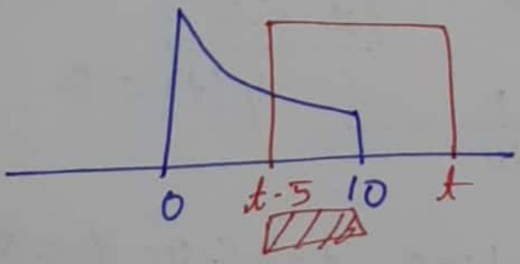
$$= \frac{1}{\log\alpha}\left[\alpha^{t} - \alpha^{t-5}\right]$$

(iii) $10 < t < 15$

$t > 10$

$(t-5 < 10 \; ; \; t < 15)$

$$y(t) = \int_{t-5}^{10} \alpha^{\tau} \cdot 1 dt$$

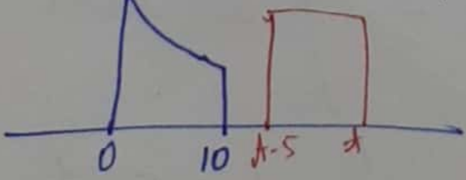


$$= \left.\frac{\alpha^{\tau}}{\log\alpha}\right]_{t-5}^{10}$$

$$= \frac{1}{\log\alpha}\left[\alpha^{10} - \alpha^{t-5}\right]$$

(iv) $t > 15$

$y(t) = 0$



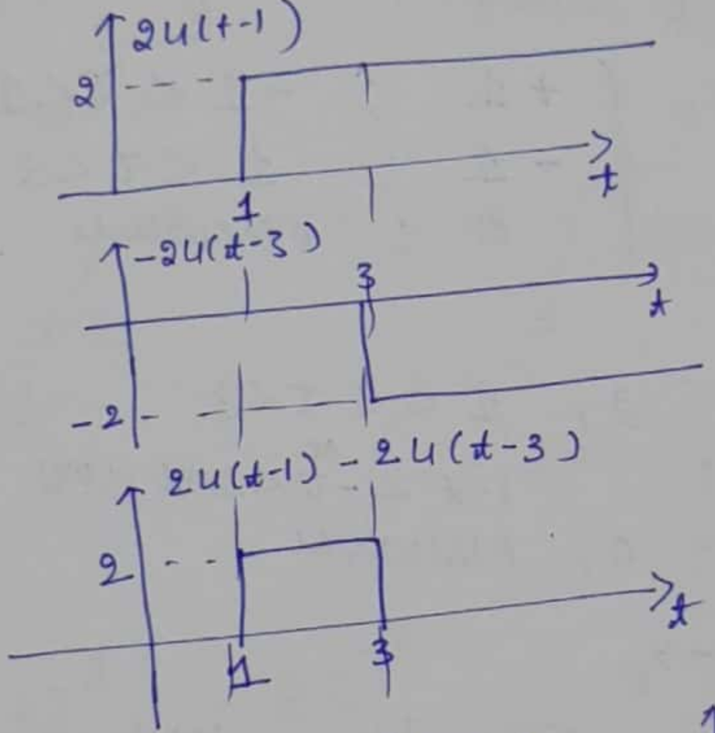
$$y(t) = \begin{cases} 0 \; ; & t < 0 \\ \frac{1}{\log\alpha}\left[\alpha^{t} - 1\right] ; & 0 < t < 5 \\ \frac{1}{\log\alpha}\left[\alpha^{t} - \alpha^{t-5}\right] ; & 5 < t < 10 \\ \frac{1}{\log\alpha}\left[\alpha^{10} - \alpha^{t-5}\right] ; & 10 < t < 15 \\ 0 \; ; & t > 15 \end{cases}$$

② $x(t) = 2u(t-1) - 2u(t-3)$

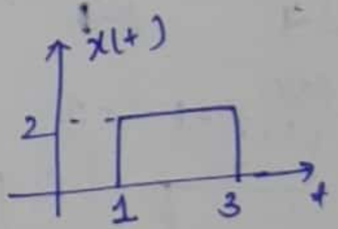
$h(t) = u(t+1) - 2u(t-1) + u(t-3)$

I Sketch the signals & redefine it

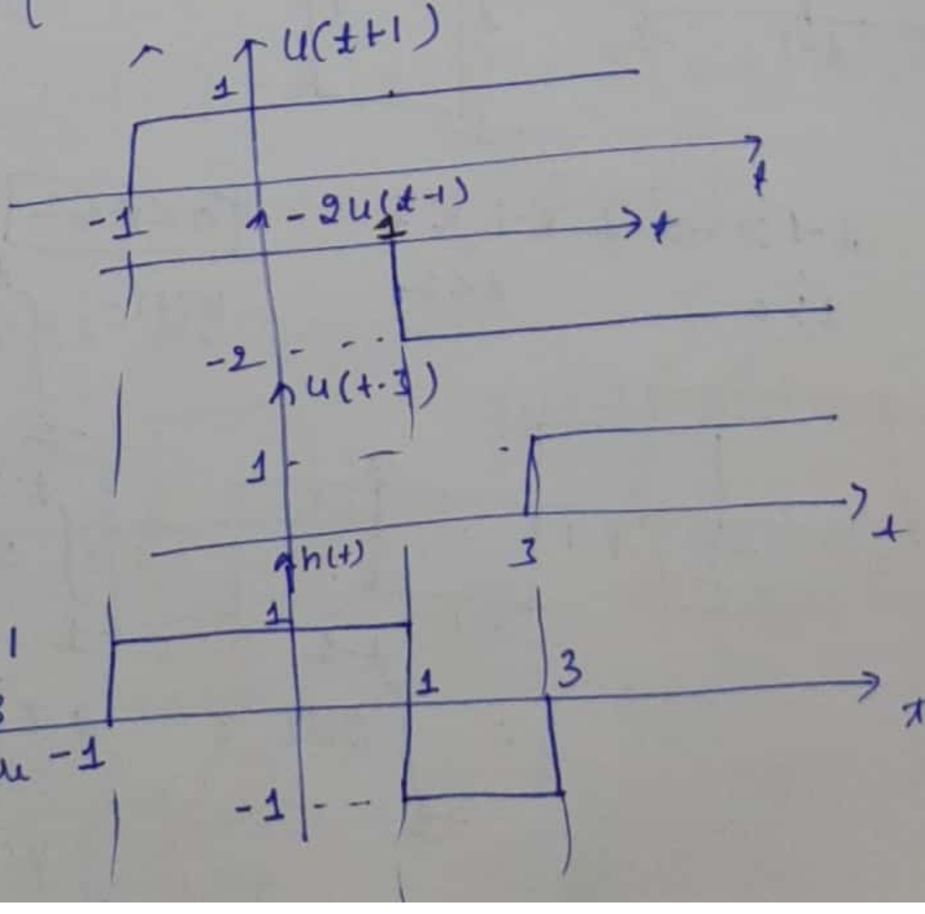
$x(t)$

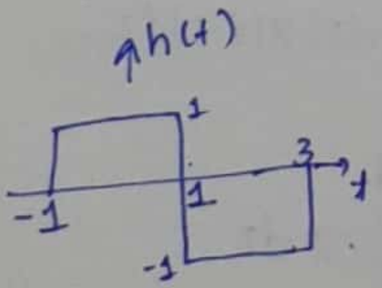


$$x(t) = \begin{cases} 2 \; ; & 1 < t < 3 \\ 0 \; ; & \text{elsewhere} \end{cases}$$



$\underline{\underline{h(t)}}$





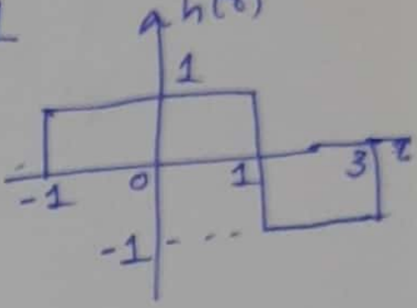
$$h(t) = \begin{cases} 1 \; ; & -1 < t < 1 \\ -1 \; ; & 1 < t < 3 \\ 0 \; ; & \text{elsewhere} \end{cases}$$

II $y(t) = h(t) * x(t)$

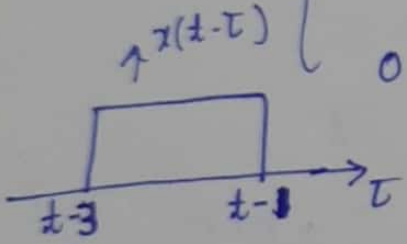
$$\triangleq \cancel{t} \int_{t=-\infty}^{\infty} h(\tau)\cdot x(t-\tau)\,d\tau$$

III Sketch & redefine the signals

$$h(\tau) = \begin{cases} +1 ; & -1 < \tau < 1 \\ -1 ; & 1 < \tau < 3 \\ 0 ; & \text{elsewhere} \end{cases}$$



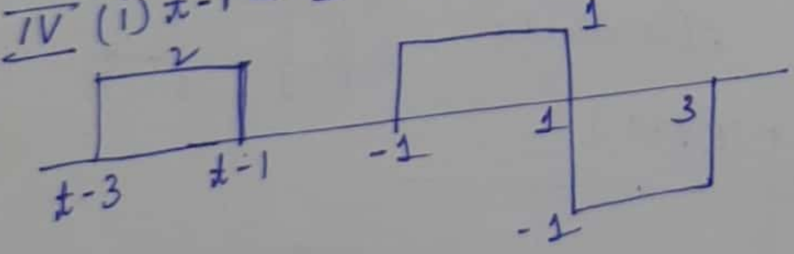
$$x(t-\tau) = \begin{cases} 2, & 1 < t-\tau < 3 \\ & \overset{\text{or}}{1-t < -\tau < 3-t} \text{ (or) } t-1 > \tau > t-3 \\ & \text{(or) } t-3 < \tau < t-1 \\ 0, & \text{elsewhere} \end{cases}$$



IV case (i) $t-1 < -1$ ∴ $\boxed{t < 0}$

$y(t) = 0$

No overlap

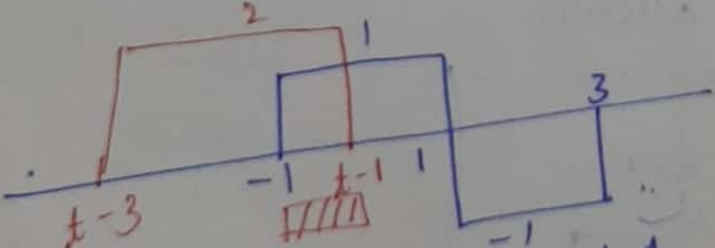


case (ii) $t-1 > -1$ & $t-1 < 1$
$\therefore t > 0$ $\quad$ $\overset{\shortparallel}{t < 2}$

$\boxed{0 < t < 2}$

$$y(t) = \int_{-1}^{t-1} h(\tau)\,x(t-\tau)\,d\tau$$

$$= \int_{-1}^{t-1} 1 \cdot 2\,d\tau$$



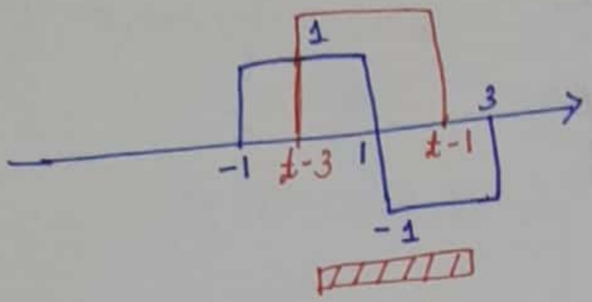
$$= \int_{-1}^{t-1} 2\,d\tau = 2\tau\Big]_{-1}^{t-1} = 2[t-1-(-1)]$$

$$= 2[t-\cancel{1}+\cancel{1}]$$

$\boxed{y(t) = 2t}$ //

case (iii) $-1 < t-3 < 1 \; = \boxed{2 < t < 4}$

$1 < t-1 < 3 \; , \; 2 < t < 4$



$$y(t) = \int_{t-3}^{t-1} h(\tau).x(t-\tau)d\tau$$

$$y(t) = \int_{t-3}^{1} h(\tau).x(t-\tau)\,d\tau + \int_{1}^{t-1} h(\tau).\,x(t-\tau)d\tau$$

$$= \int_{t-3}^{1} 1.2\,d\tau + \int_{1}^{t-1} -1.2d\tau$$

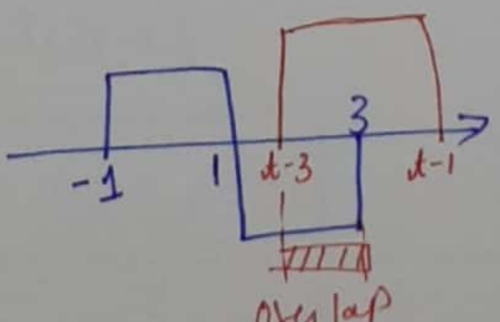
$$= 2\tau\Big]_{t-3}^{1} - 2\tau\Big]_{1}^{t-1}$$

$$= 2\left[1-(t-3)\right] - 2\left[t-1-1\right]$$

$$= 2\left[1-t+3\right] - 2\left[t-2\right] = 2\left[4-t\right] - 2\left[t-2\right]$$

$$= 8-2t-2t+4 = \boxed{12-4t}$$

case (iv) $1 < t-3 < 3 \Rightarrow \boxed{4 < t < 6}$



$$y(t) = \int_{t-3}^{3} h(\tau).x(t-\tau)d\tau$$

$$= \int_{t-3}^{3} -1.2d\tau = -2\tau\Big]_{t-3}^{3}$$

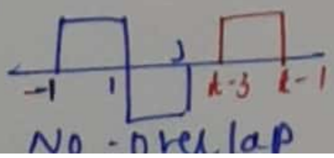
$$= -2\left[3-t+3\right] = -2\left[6-t\right]$$

$$= \boxed{2t-12} \; ,,$$

case (v) $t-3 > 3$

$t > 6$

$\boxed{y(t) = 0}$

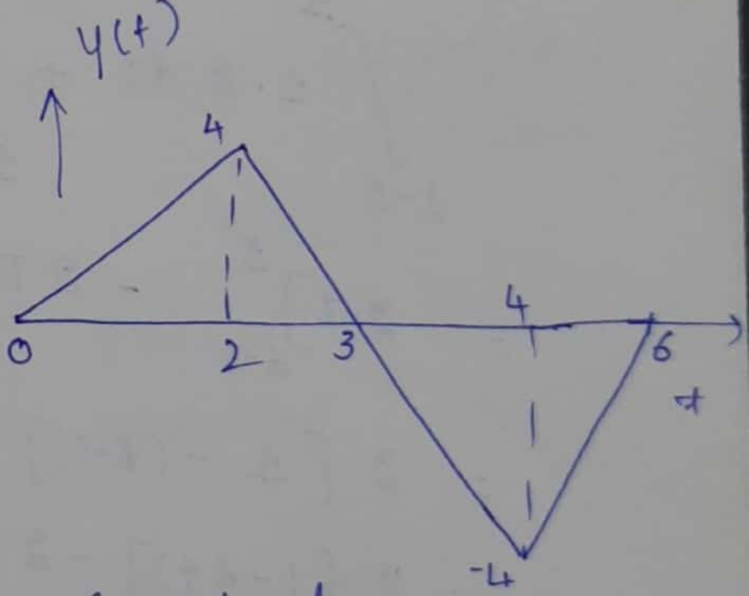


No-overlap

$$y(t) = \begin{cases} 0 & ; t < 0 \\ 2t & ; 0 < t < 2 \\ 12-4t & ; 2 < t < 4 \\ 2t-12 & ; 4 < t < 6 \\ 0 & ; t > 6 \end{cases}$$

* Calculate the Values of $y(t)$ for t

t	$y(t)$
0	0
1	2
1.5	3
1.9	3.8
2	4
2.5	2
2.9	0.4
3	0
3.5	-2
3.9	-3.6
4	-4
4.5	-3
4.9	-2.2
5	-2
5.5	-1
5.9	-0.2
6	0

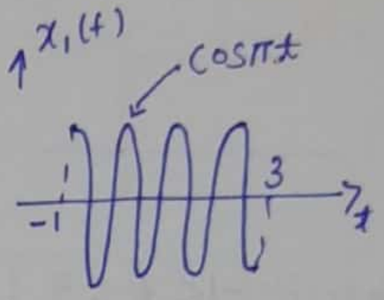


Convoluted Output signal

3. $x_1(t) = \cos\pi t\,[u(t+1) - u(t-3)]$

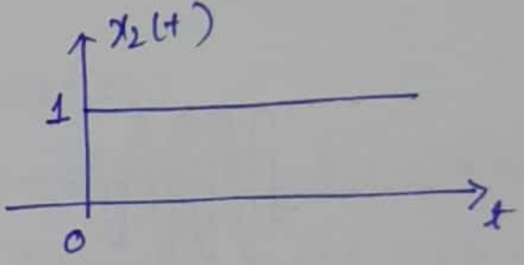
$x_2(t) = u(t)$

I) $x_1(t) = \begin{cases} \cos\pi t & ; -1 < t < 3 \\ 0, & \text{elsewhere} \end{cases}$



$u(t+1) - u(t-3) = \begin{cases} 1, & -1 < t < 3 \\ 0, & \text{elsewhere} \end{cases}$

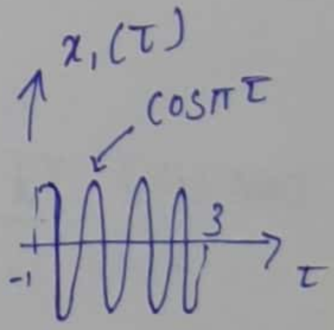
$x_2(t) = \begin{cases} 1; & t > 0 \\ 0, & t < 0 \end{cases}$



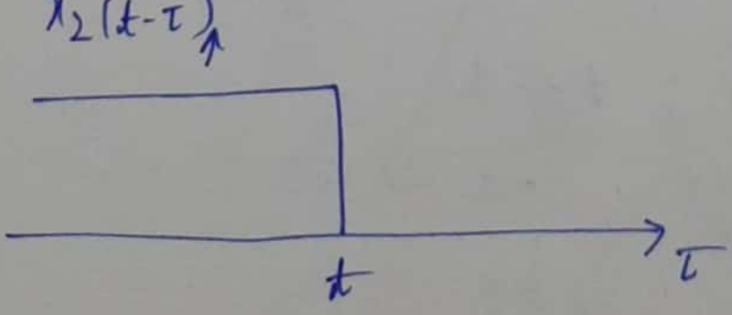
II $y(t) = x_1(t) * x_2(t)$

$\triangleq \int_{\tau=-\infty}^{\infty} x_1(\tau)\cdot x_2(t-\tau)\,d\tau$

III $x_1(\tau) = \begin{cases} \cos\pi\tau & ; -1 < \tau < 3 \\ 0 & ; \text{elsewhere} \end{cases}$



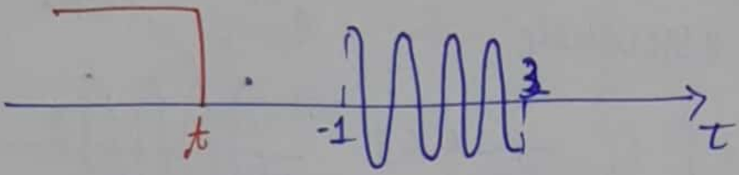
$x_2(t-\tau) = \begin{cases} 1 & ; t-\tau > 0 \text{ or } -\tau > -t \\ & \text{or } \boxed{\tau < t} \\ 0 & ; \text{elsewhere} \end{cases}$



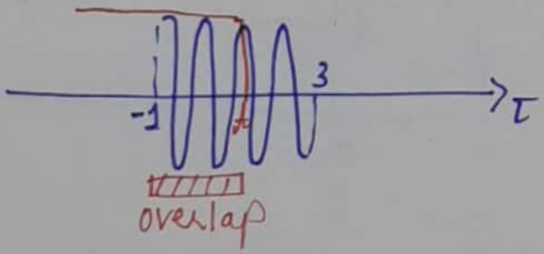
IN

case(i) $\boxed{t < -1}$

No overlap $\therefore y(n) = 0$



case(ii) $\boxed{-1 < t < 3}$



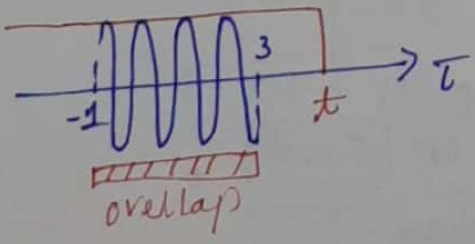
$$y(t) = \int_{-1}^{t} x_1(\tau) \cdot x_2(t-\tau) d\tau$$

$$= \int_{-1}^{t} \cos\pi\tau \cdot 1 \, d\tau$$

$$= \left.\frac{\sin\pi\tau}{\pi}\right]_{-1}^{t} = \underline{\underline{\frac{\sin\pi t}{\pi}}}$$

case (iii) $\boxed{t > 3}$

$$y(t) = \int_{-1}^{3} \cos\pi\tau \cdot 1 \, d\tau$$



$$= \left.\frac{\sin\pi\tau}{\pi}\right]_{-1}^{3}$$

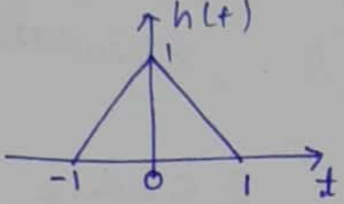
$$= \underline{\underline{0}}$$

$$y(t) = \begin{cases} 0 \; ; & t < -1 \\ \frac{\sin\pi t}{\pi} \; ; & -1 < t < 3 \\ 0 \; ; & t > 3 \end{cases}$$

④ Perform the convolution between the following signals ⑭ & sketch the convoluted o/p

$x(t) = \delta(t) + 2\,\delta(t-2) - 10\,\delta(t-3)$ $h(t) = \begin{cases} 1+t\ ; & -1<t<0 \\ 1-t\ ; & 0<t<1 \end{cases}$

I Sketch only h(t)



II WKT $x(t) * \delta(t) = x(t)$

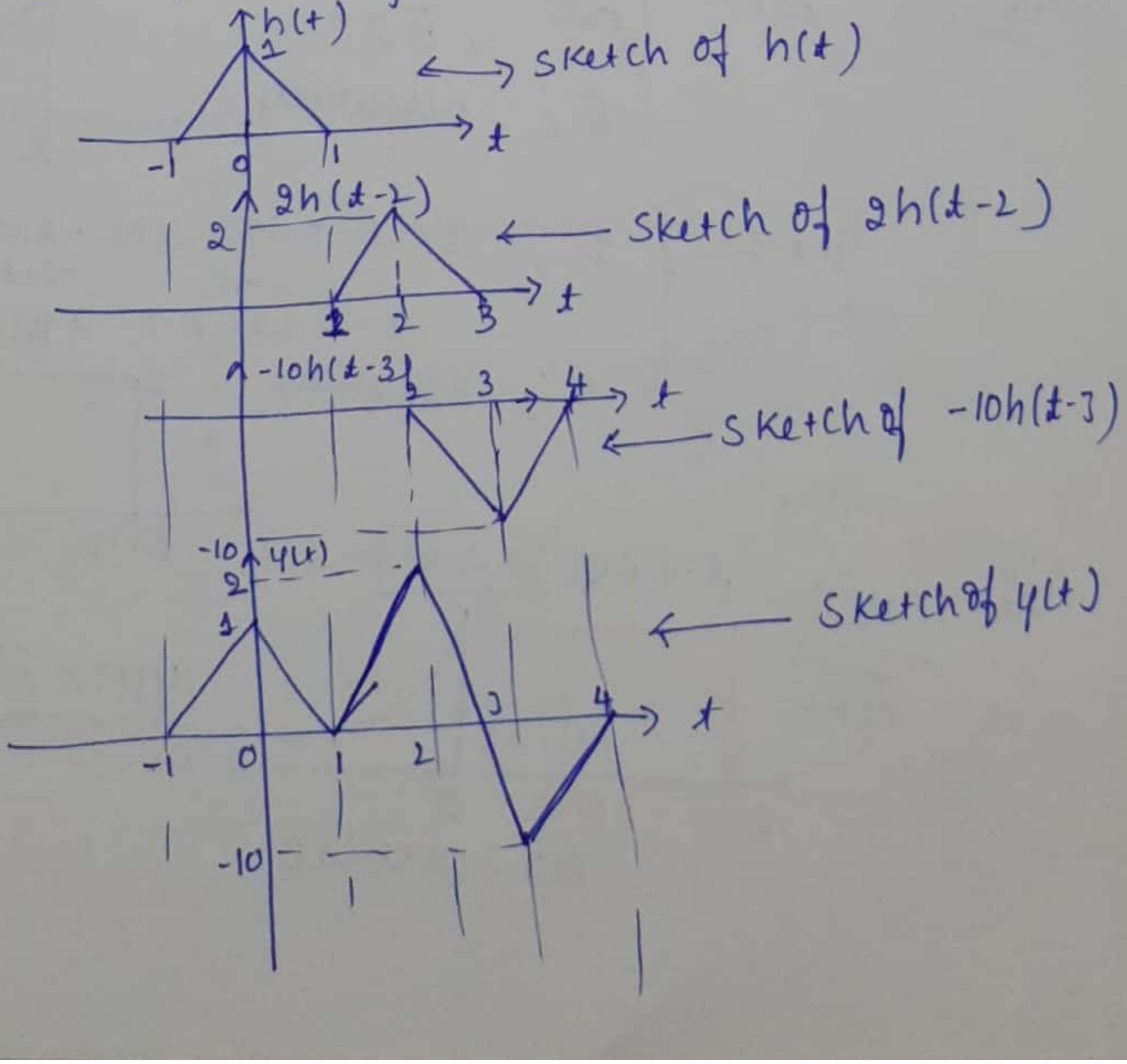
& $x(t) * \delta(t-t_0) = x(t-t_0)$

$y(t) = x(t) * h(t) = [\delta(t) + 2\,\delta(t-2) - 10\,\delta(t-3] * h(t)$

$= \delta(t) * h(t) + 2\,\delta(t-2) * h(t) - 10\,\delta(t-3) * h(t)$

$$\boxed{y(t) = h(t) + 2h(t-2) - 10\,h(t-3)}$$

III Sketch y(t) using the above eqn



⟷ sketch of h(t)

← sketch of 2h(t-2)

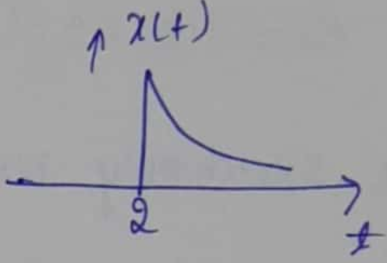
← sketch of -10h(t-3)

← sketch of y(t)

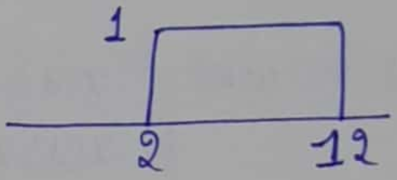
⑤ $x(t) = e^{-2t} u(t-2)$

$h(t) = u(t-2) - u(t-12)$

I $x(t) = \begin{cases} e^{-2t} & ; t > 2 \\ 0 & ; \text{elsewhere} \end{cases}$



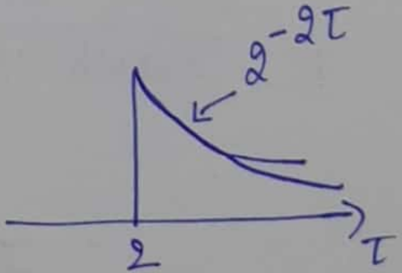
$h(t) = \begin{cases} 1 & ; 2 < t < 12 \\ 0 & ; \text{elsewhere} \end{cases}$



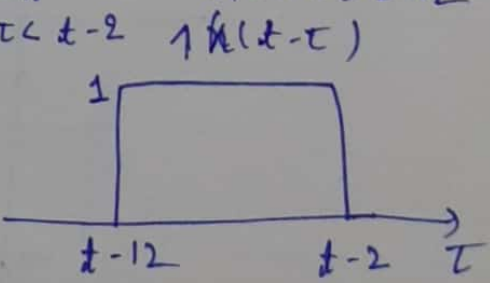
II $y(t) = x(t) * h(t)$

$$= \int_{\tau=-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\, d\tau$$

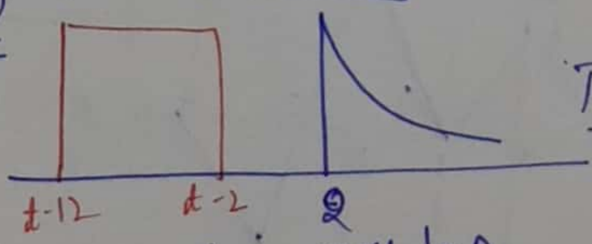
III $x(\tau) = \begin{cases} e^{-2\tau} & ; \tau > 2 \\ 0 & ; \text{elsewhere} \end{cases}$



$h(t-\tau) = \begin{cases} 1 ; & 2 < t-\tau < 12 \text{ (or) } -t+2 < -\tau < 12-t \\ & -2+t > \tau > t-12 \\ & \text{(or)} \\ & t-12 < \tau < t-2 \\ 0 ; & \text{elsewhere} \end{cases}$



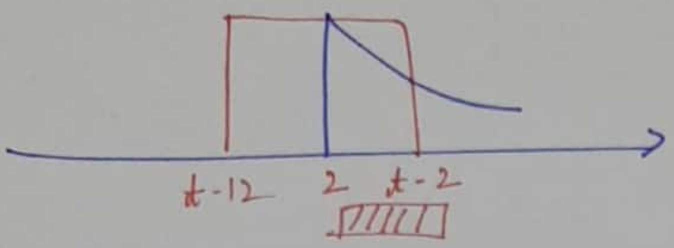
IV Case (i) $t-2 < 2$ $\boxed{t < 4}$



$\boxed{y(t) = 0}$

No - overlap

case (ii) $t-2>2$ $t>4$ $\boxed{4<t<14}$



$$y(t) = \int_{2}^{t-2} e^{-2\tau}\cdot 1\, d\tau$$

$$= \int_{2}^{t-2} e^{-2\tau} d\tau$$

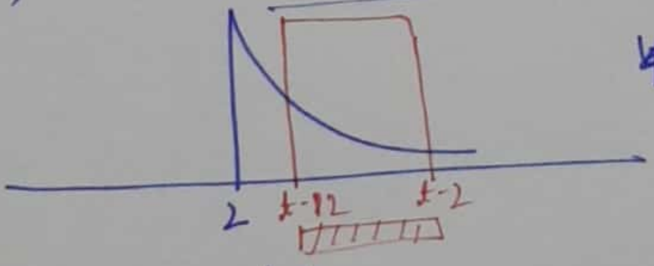
$$= \left.\frac{e^{-2\tau}}{-2}\right]_{0}^{t-2}$$

$$\boxed{y(t) = \begin{cases} 0, & t < 4 \\ \frac{1-e^{-2t+4}}{2}, & 4 < t < 14 \\ \frac{e^{-2t+24}-e^{-2t+4}}{2}; & t>14 \end{cases}} =$$

$$= \frac{e^{-2(t-2)} - e^{0}}{-2}$$

$$= \frac{1 - e^{-2t+4}}{2}$$

case (iii) $t-12>2$ $\boxed{t>14}$



$$y(t) = \int_{t-12}^{t-2} e^{-2\tau} d\tau = \left.\frac{e^{-2\tau}}{-2}\right]_{t-12}^{t-02}$$

$$= \frac{e^{-2(t-02)} - e^{-2(t-12)}}{-2}$$

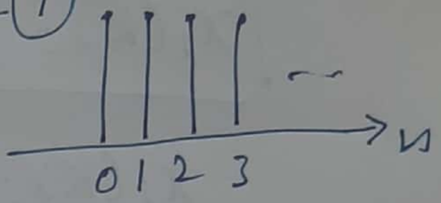
$$= -\frac{e^{-2t+4} - e^{-2t+24}}{2} = \frac{e^{-2t+24} - e^{-2t+4}}{2}$$

$x_1(n) = a^n u(n)$ $x_2(n) = a^{-n} u(-n)$ find convolution of $x_1(n) * x_2(n)$

I define & sketch the signal

$x_1(n) = a^n u(n)$

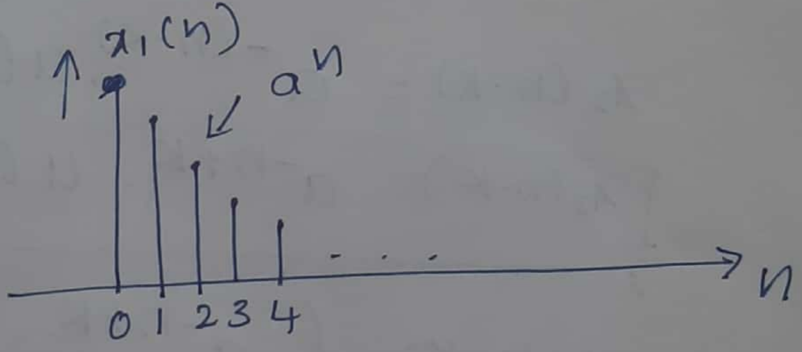
WKT $u(n) = \begin{cases} 1 ; & n \geq 0 \\ 0 ; & n < 0 \end{cases}$ — ①



$\therefore a^n u(n) = \begin{cases} a^n ; & n \geq 0 \\ 0 ; & n < 0 \end{cases}$

amplitude

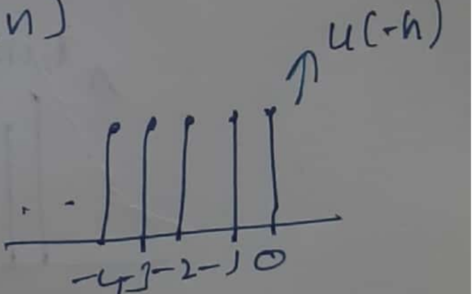
assume $0 < a < 1$



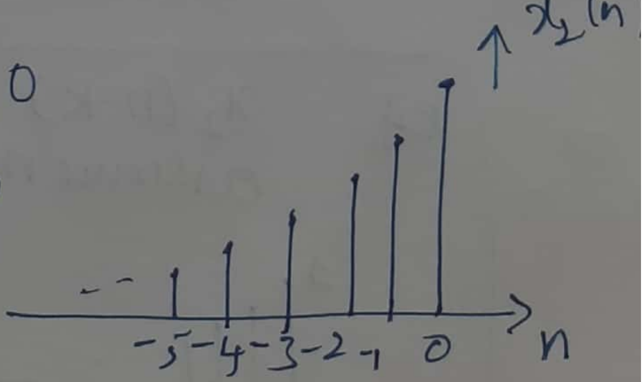
$x_2(n) = a^{-n} u(-n)$

$u(-n) =$ reflection of $u(n)$

$= \begin{cases} 1 ; & n \leq 0 \\ 0 ; & n \geq 0 \end{cases}$



$\therefore a^{-n} u(-n) = \begin{cases} a^{-n} ; & n \leq 0 \\ 0 ; & n > 0 \end{cases}$

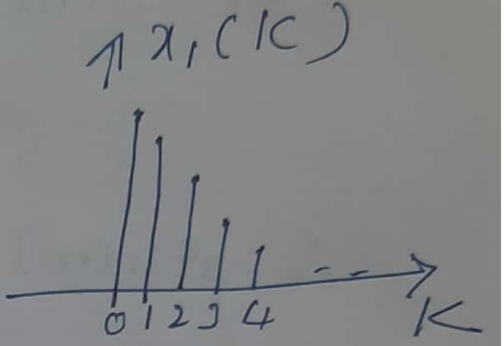


II $y(n) = x_1(n) * x_2(n)$

$$\triangleq \sum_{K=-\infty}^{\infty} x_1(K) \cdot x_2(n-K).$$

III define & sketch $x_1(K)$ & $x_2(n-k)$

$$\boxed{x_1(K) = a^K u(K)}$$

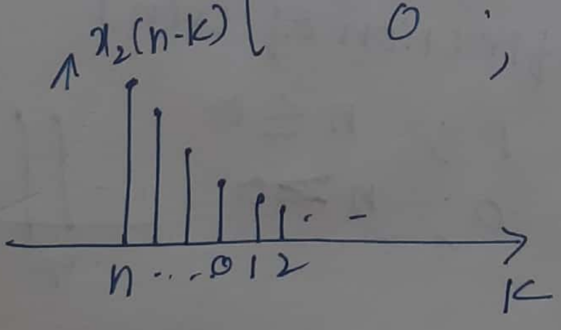


$$x_1(K) = \begin{cases} a^K ; & K \ge 0 \\ 0 ; & K < 0 \end{cases}$$

$$x_2(n-K) = a^{-(n-K)} \cdot u(-(n-K))$$

$$\boxed{x_2(n-K) = a^{-n+K} . u(-n+K)}$$

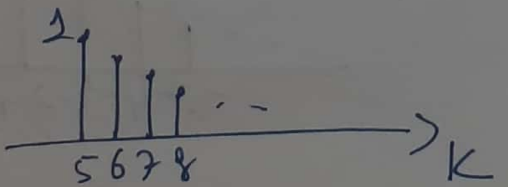
$$x_2(n-K) = \begin{cases} a^{-n+K} ; & n-K \le 0 \text{ (or)} \\ & -K \le -n \text{ (or)} \\ & \boxed{K \ge n} \\ 0 ; & K < n \end{cases}$$



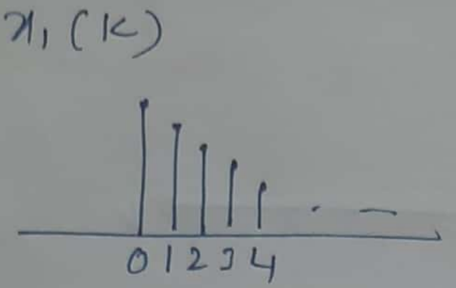
~~eg.~~

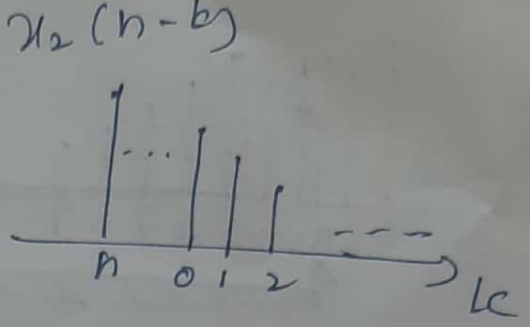
eg. $x_2(n-k)$
assume $n = 5$ then

$$x_2(5-K) = \begin{cases} a^{-5+K} ; & K \ge 5 \\ 0 ; & K < 5 \end{cases}$$



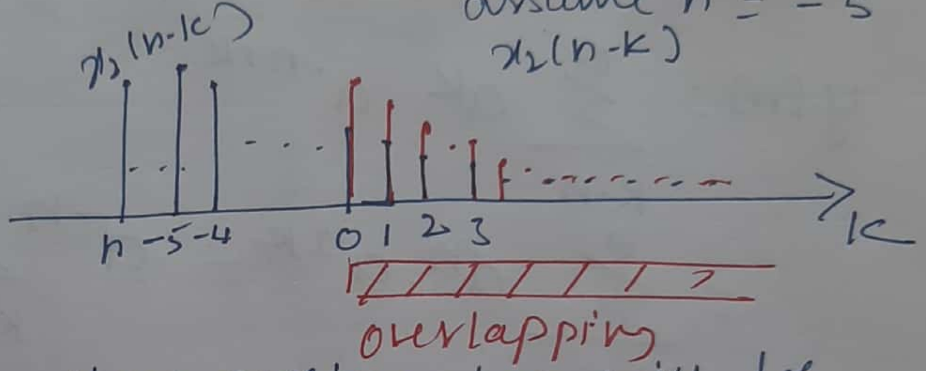
IV to find y(n) for different cases





(i) $\boxed{n<0}$

assume $n=-5$



so if
n is < 0 then overlapping will be only from $k=0$ to ∞

$$\therefore \; y(n) = \sum_{k=0}^{\infty} a^k \cdot \underbrace{a^{-n+k}}_{a^{-n}.a^k}$$

$$= \sum_{k=0}^{\infty} a^{-n}.\underbrace{a^k.a^k}_{a^{2k}}$$

$$= a^{-n}\sum_{k=0}^{\infty} a^{2k} = a^{-n}\sum_{k=0}^{\infty}(a^2)^k$$

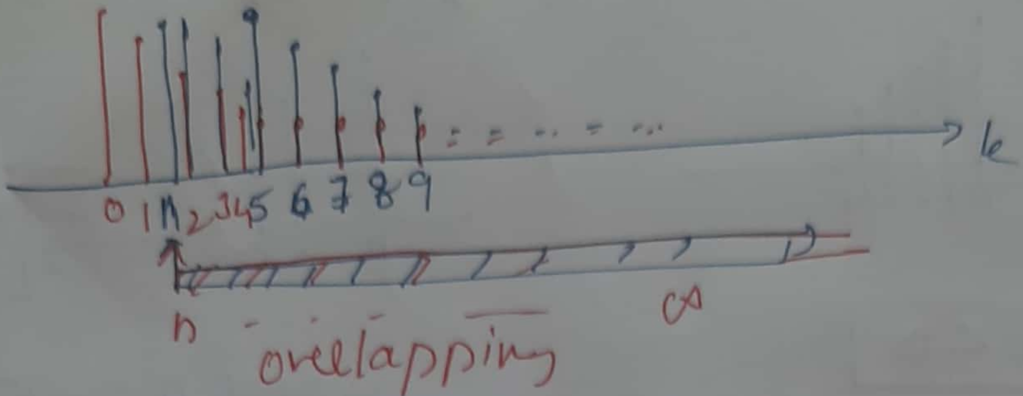
$$\boxed{\text{WKT} \sum_{k=0}^{\infty} a^k = \frac{1}{1-a}}$$

here
$a = a^2$

$$= a^{-n}\left[\frac{1}{1-a^2}\right] = \frac{a^{-n}}{1-a^2}$$

(ii) $n \geq 0$

assume $n=5$ then



$$y[n] = \sum_{k=n}^{\infty} a^k . a^{-n+k}$$

$$= \sum_{k=n}^{\infty} a^{-n} . a^{2k} = a^{-n} \sum_{k=n}^{\infty} (a^2)^k$$

$$\boxed{\text{wkt} \sum_{k=n}^{\infty} a^k = \frac{a^n}{1-a}}$$ here $a = a^2$

$$\therefore y(n) = \frac{a^{-n} . (a^2)^n}{1-a^2} = \frac{a^{-n} a^{2n}}{1-a^2}$$

$$= \frac{a^{-n+2n}}{1-a^2}$$

$$= \frac{a^n}{1-a^2}$$

$$\boxed{y(n) = \begin{cases} \frac{a^{-n}}{1-a^2} & ; n \leq 0 \\ \frac{a^n}{1-a^2} & ; n > 0 \end{cases}}$$

② $x_1(n) = \{ \underset{\uparrow}{2}, 1 \}$ $x_2(n) = \{ 1, \underset{\uparrow}{2}, 3 \}$

convolute by defⁿ

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 2 & 2 & 4 & 6 \\ 1 & 1 & 2 & 3 \end{array}$$

$y(n) = \{ 2, \underset{\uparrow}{5}, 8, 3 \}$

$$y[n] = x_1[n] * x_2[n]$$

$$\triangleq \sum_{K=-\infty}^{\infty} x_1(K) \cdot x_2(n-K)$$

$\boxed{n=0}$ $$y(0) = \sum_{K=-\infty}^{\infty} x_1(K) \cdot x_2(-K)$$

$$\begin{array}{lccc} x_1(K) \rightarrow & & \overset{\downarrow}{2} & 1 \\ x_2(-K) \rightarrow & 3 & 2 & 1 \\ \hline y(0) = & & 4 + 1 & = 5 \end{array}$$

$\boxed{y(0) = 5}$

$\boxed{n=1}$ $$y(1) = \sum_{K=-\infty}^{\infty} x_1(K) \, x_2(1-K)$$

$$\begin{array}{lccc} x_1(k) \rightarrow & \overset{\downarrow}{2} & 1 & \\ x_2(1-K) \rightarrow & 3 & 2 & 1 \\ \hline & 6 + 2 & & = 8 \end{array}$$

$\boxed{y(1) = 8}$

$\boxed{n=2}$ $$y(2) = \sum_{K=-\infty}^{\infty} x_1(K) \, x_2(2-K)$$

$$\begin{array}{lcccc} x_1(K) \rightarrow & \overset{\downarrow}{2} & 1 & & \\ x_2(2-K) \rightarrow & & 3 & 2 & 1 \\ \hline & & 3 & & \end{array}$$

$\boxed{y(2) = 3}$

$n = 3 \quad y(3) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(3-k)$

$x_1(k) \qquad \overset{\downarrow}{2} \; 1$

$x_2(3-k) \qquad\qquad 3\;2\;1 \qquad \boxed{y(3) = 0}$

no overlap

$n = -1 \quad y(-1) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(-1-k)$

$x_1(k) = \quad \overset{\downarrow}{2} \;\; 1$

$x_2(-1-k) = 3\;2\;1$

$2 \qquad \boxed{y(-1) = 2}$

$n = -2 \quad y(-2) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(-2-k)$

$x_1(K) = \quad \overset{\downarrow}{2} \;\; 1$

$x_2(-2-k) \; 3\;2\;1$

no overlap

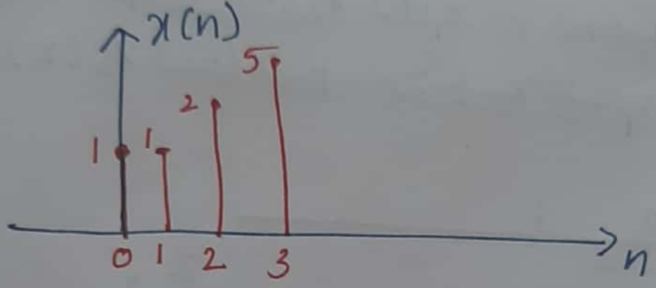
$y(n) = \{2, \underset{\uparrow}{5}, 8, 3,\}.$

③ - A discrete-time LTI system is characterized by the impulse response $h[n] = u[n]$
find its output when the input is

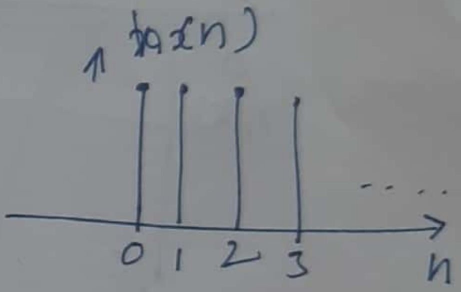
$$x[n] = \{1, 1, 2, 5\}$$

also find $y(-19)$, $y(0)$, $y(2)$, $y(15)$ & $y(28)$

(I) $x[n] = \{1, 1, 2, 5\}$



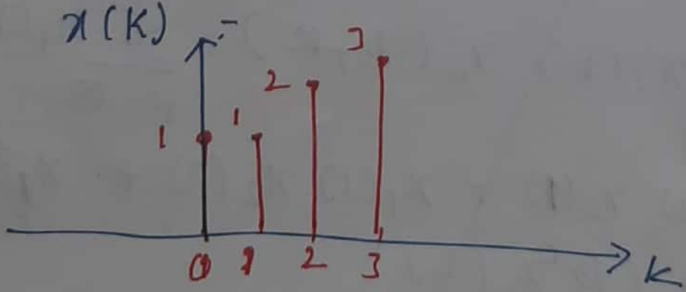
$$h(n) = u(n) = \begin{cases} 1; & n \geq 0 \\ 0; & n < 0 \end{cases}$$



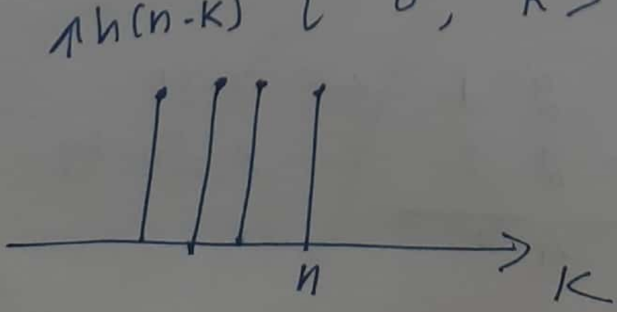
II $y(n) = x(n) * h(n)$

$$\triangleq \sum_{K=-\infty}^{\infty} x(K) \cdot h(n-K).$$

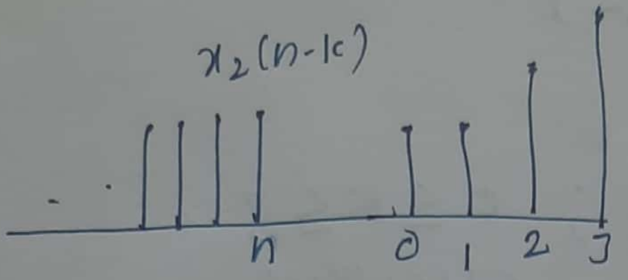
III



$$h(n-K) = \begin{cases} 1, & n-K \geq 0 \text{ (or) } K \leq n \\ 0; & K > n \end{cases}$$



(i) for $n < 0$



$y(n) = 0$

(ii) $n = 0$; $y(0) = 1 \cdot 1 = 1$

$n = 1$ $\quad y(1) = 1 \cdot 1 + 1 \cdot 1 = 2$

$n = 2$ $\quad y(2) = 1 \cdot 1 + 1 \cdot 1 + 1 . 2 = 4$

$n = 3$ $\quad y(3) = 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 3 = 9$

$n = 4$ $\quad y(4) = 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 3 = 9$

$\vdots$

$$y[n] = \{ \underset{\uparrow}{1}, 2, 4, 9, 9, 9 . \; . \; - \}$$

$$\boxed{\begin{array}{lll} y[-19] = 0 & y(2) = 4 & \\ y(0) = 1 & y(15) = 9 & y(28) = 9 \end{array}}$$

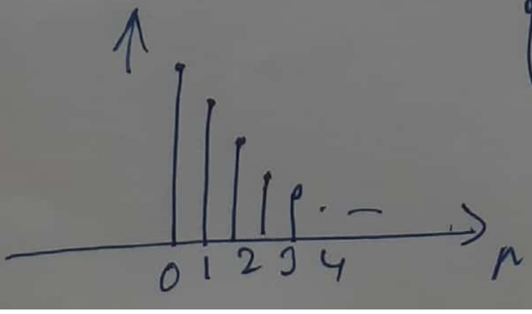
④ $x(n) = n + 2 ; \quad 0 \leq n \leq 3$

$h(n) = a^n u(n)$

I $\quad x(n) = \{ \underset{\uparrow}{2}, 3, 4, 5 \}$

n	n+2
0	2
1	3
2	4
3	5

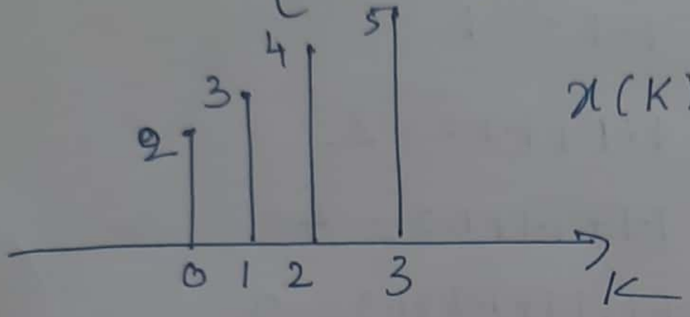
$$h(n) = a^n u(n) = \begin{cases} a^n ; & n \geq 0 \\ 0 ; & n < 0 \end{cases}$$



II $y(n) = x(n) * h(n)$

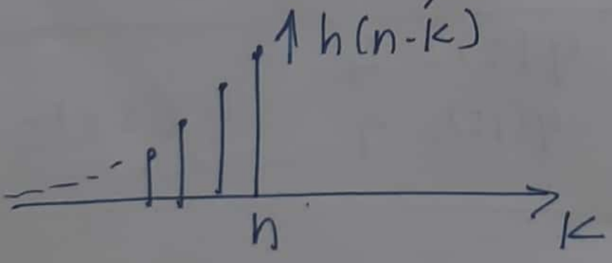
$$\triangleq \sum_{K=-\infty}^{\infty} x(K)\cdot h(n-K)$$

III $x(K) = \begin{cases} K+2 ; & 0 \le K \le 3 \\ 0 ; & \text{elsewhere} \end{cases}$



$x(K) = \{\underset{\uparrow}{2}, 3, 4, 5\}$

$h(n-K) = \begin{cases} a^{n-K} ; & n-K \ge 0 \text{ or } -K \ge -n \\ & \boxed{K \le n} \\ 0 ; & K > n \end{cases}$



IV (i) $n < 0$ $\quad y(n) = \underline{\underline{0}}$

(ii) ?

solve it

show that

a) $x(n) * \delta(n) = x(n)$

$$x(n) * \delta(n) \triangleq \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$= x(k)\Big|_{k=n}$$ [using shifting property]

$$= x(n)$$

(b) $x(n) * \delta(n-n_0) = x(n-n_0)$

$$x(n) * \delta(n-n_0) = \sum_{k=-\infty}^{\infty} x(k)\, \delta(n-n_0-k)$$

$$= x(k)\Big|_{k=n-n_0}$$ [using shifting property]

$$= x(n-n_0)$$

(c) $x(n) * u(n) = \sum_{k=-\infty}^{n} x(k)$

LHS $x(n) * u(n) = \sum_{k=-\infty}^{\infty} x(k)\, u(n-k)$

we have $u(n-k) = 1$; when $n-k \geq 0$ $k \leq n$

0 ; when $n-k < 0$ $k > n$

$$x(n) * u(n) = \sum_{k=-\infty}^{n} x(k)$$

(d) $x(n) * u(n-no) = \sum_{K=-\infty}^{n-no} x(K)$

LHS

$$x(n) * u(n-no) \triangleq \sum_{K=-\infty}^{\infty} x(K)\, u(n-n_0-K)$$

we have

$u(n-n_0-K) = 1$; when

$n-no-K \geq 0$

$K \leq n-no$

$= 0$; when $K > n-no$

$$x(n) * u(n-no) = \sum_{K=-\infty}^{n-no} x(K)$$

(e) ~~$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)$~~ $d\tau$

LHS $x(t)$

(e) $x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau) . \delta(t-\tau) d\tau$

$= x(\tau)\big|_{\tau=t}$ [using shifting property]

$= \underline{\underline{x(t)}}$

(f) $x(t) * \delta(t-t_0) = x(t-t_0)$

LHS

$$x(t) * \delta(t-t_0) = \int_{-\infty}^{\infty} x(\tau) \cdot \delta(t-t_0-\tau) d\tau$$

$$= x(\tau) \Big|_{\tau = t-t_0}$$ [using Shifting Property]

$$= x(t-t_0)$$

(g) $$x(t) * u(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

WKT $u(t-\tau) = 1$; when $t-\tau \geq 0$
ie, $\tau \leq t$
$= 0$; when $t-\tau < 0$
ie, $\tau > t$

$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

(h) $$x(t) * u(t-t_0) = \int_{-\infty}^{t-t_0} x(\tau) d\tau$$

$u(t-t_0-\tau) = 1$; when $t-t_0-\tau \geq 0$ $\tau \leq t-t_0$
$= 0$; $\tau > t-t_0$

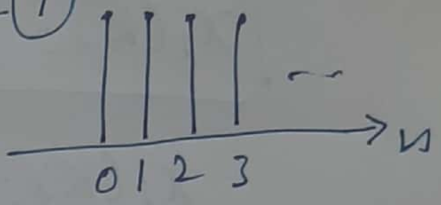
$$x(t) * u(t-t_0) = \int_{-\infty}^{t-t_0} x(\tau) d\tau$$

$x_1(n) = a^n u(n)$ $x_2(n) = a^{-n} u(-n)$ find convolution of $x_1(n) * x_2(n)$

I define & sketch the signal

$x_1(n) = a^n u(n)$

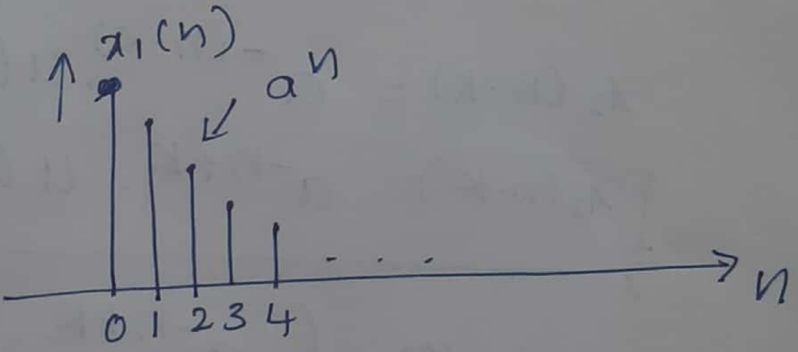
WKT $u(n) = \begin{cases} 1 ; & n \geq 0 \\ 0 ; & n < 0 \end{cases}$ — ①



$\therefore\ \overset{\uparrow}{a^n}\, u(n) = \begin{cases} a^n ; & n \geq 0 \\ 0 ; & n < 0 \end{cases}$

amplitude

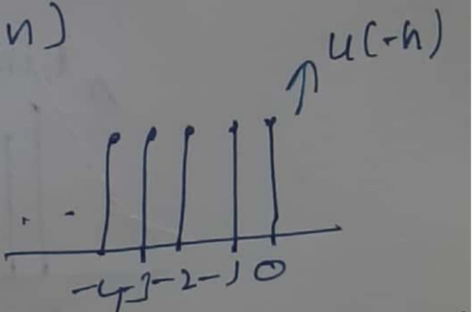
assume
$0 < a < 1$



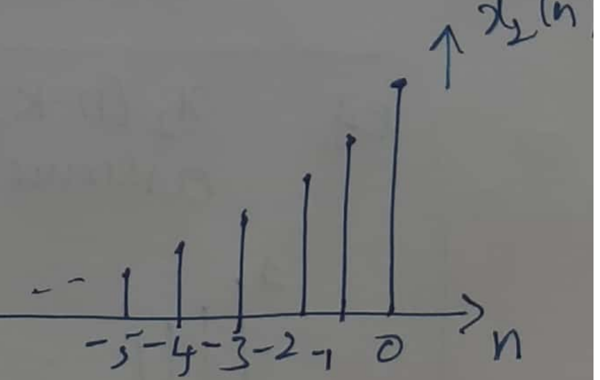
$x_2(n) = a^{-n} u(-n)$

$u(-n)$ = reflection of $u(n)$

$= \begin{cases} 1 ; & n \leq 0 \\ 0 ; & n \geq 0 \end{cases}$



$\therefore\ a^{-n} u(-n) = \begin{cases} a^{-n} ; & n \leq 0 \\ 0 ; & n > 0 \end{cases}$



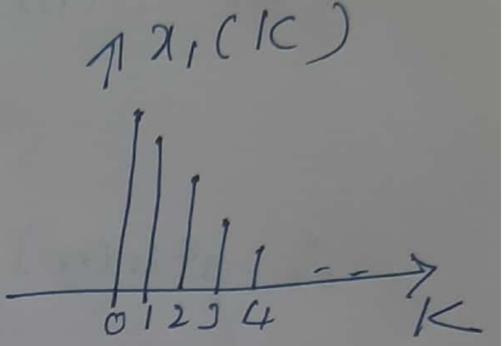
II $y(n) = x_1(n) * x_2(n)$

$$\triangleq \sum_{k=-\infty}^{\infty} x_1(k) \cdot x_2(n-k).$$

III define & sketch $x_1(k)$ & $x_2(n-k)$

$$\boxed{x_1(k) = a^k u(k)}$$

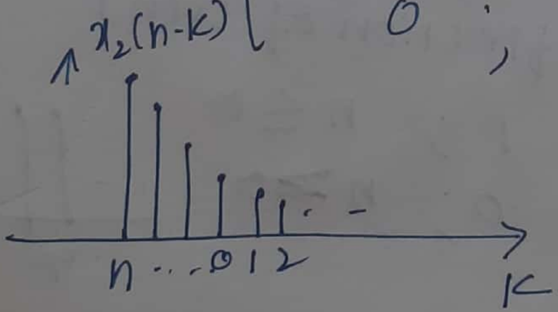
$$x_1(k) = \begin{cases} a^k ; & k \geq 0 \\ 0 ; & k < 0 \end{cases}$$



$$x_2(n-k) = a^{-(n-k)} . u(-(n-k))$$

$$\boxed{x_2(n-k) = a^{-n+k} . u(-n+k)}$$

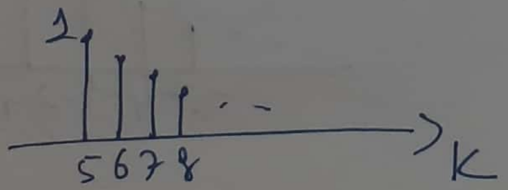
$$x_2(n-k) = \begin{cases} a^{-n+k} ; & n-k \leq 0 \text{ (or) } -k \leq -n \text{ (or) } \boxed{k \geq n} \\ 0 ; & k < n \end{cases}$$



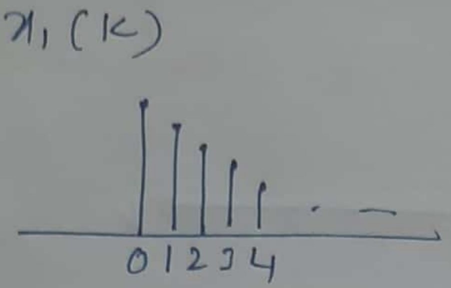
~~eg.~~

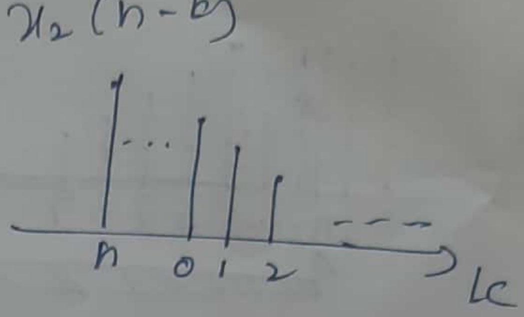
eg. $x_2(n-k)$
assume $n=5$

then $x_2(5-k) = \begin{cases} a^{-5+k} ; & k \geq 5 \\ 0 ; & k < 5 \end{cases}$



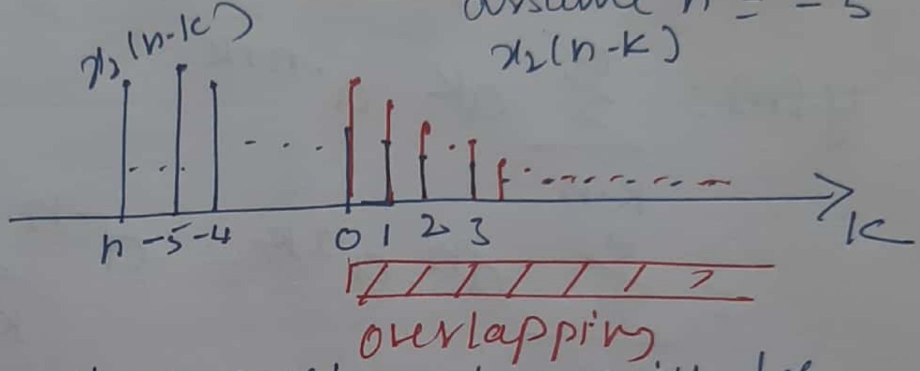
IV to find $y(n)$ for different cases





(i) $\boxed{n < 0}$

assume $n = -5$



so if n is < 0 then overlapping will be only from $K = 0$ to ∞

$$\therefore\ y(n) = \sum_{K=0}^{\infty} a^K \cdot \underbrace{a^{-n+K}}_{a^{-n}.a^K}$$

$$= \sum_{K=0}^{\infty} a^{-n}.\underbrace{a^K.a^K}_{a^{2k}}$$

$$= a^{-n} \sum_{K=0}^{\infty} a^{2K} = a^{-n} \sum_{K=0}^{\infty} (a^2)^K$$

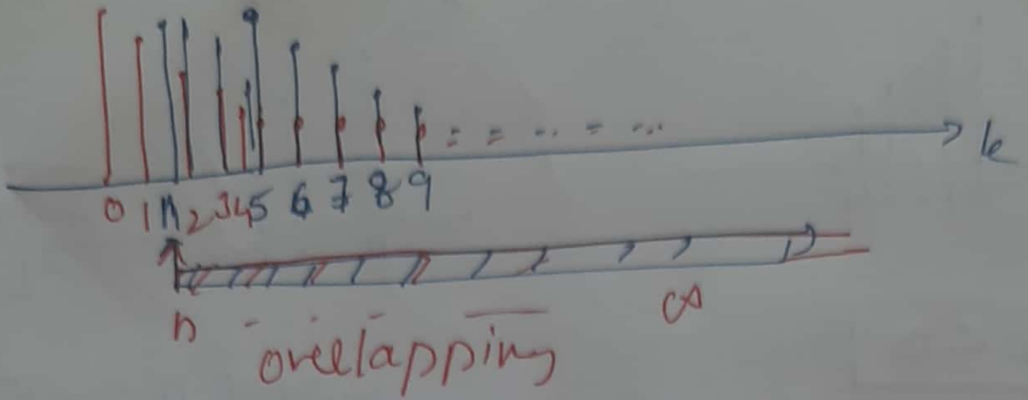
$$\boxed{\text{WKT} \sum_{K=0}^{\infty} a^k = \frac{1}{1-a}}$$

here $a = a^2$

$$= a^{-n}\left[\frac{1}{1-a^2}\right] = \frac{a^{-n}}{1-a^2}$$

(ii) $n \geq 0$

assume $n = 5$ then



$$y(n) = \sum_{k=n}^{\infty} a^{k} \cdot a^{-n+k}$$

$$= \sum_{k=n}^{\infty} a^{-n} \cdot a^{2k} = a^{-n} \sum_{k=n}^{\infty} (a^{2})^{k}$$

$$\boxed{\text{wkt} \quad \sum_{k=n}^{\infty} a^{k} = \frac{a^{n}}{1-a}}$$

her $a = a^{2}$

$$\therefore \; y(n) = \frac{a^{-n} \cdot (a^{2})^{n}}{1-a^{2}} = \frac{a^{-n} \, a^{2n}}{1-a^{2}}$$

$$= \frac{a^{-n+2n}}{1-a^{2}}$$

$$= \frac{a^{n}}{1-a^{2}}$$

$$\boxed{y(n) = \begin{cases} \dfrac{a^{-n}}{1-a^{2}} & ; \; n \leq 0 \\ \dfrac{a^{n}}{1-a^{2}} & ; \; n > 0 \end{cases}}$$

② $x_1(n) = \{ \underset{\uparrow}{2}, 1 \}$ $x_2(n) = \{ 1, \underset{\uparrow}{2}, 3 \}$

convolute by def^n

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 2 & 2 & 4 & 6 \\ 1 & 1 & 2 & 3 \end{array}$$

$y(n) = \{ 2, \underset{\uparrow}{5}, 8, 3 \}$

$y[n] = x_1[n] * x_2[n]$

$\triangleq \sum_{K=-\infty}^{\infty} x_1(K) \cdot x_2(n-K)$

$\boxed{n=0}$ $y(0) = \sum_{K=-\infty}^{\infty} x_1(K) \cdot x_2(-K)$

$$\begin{array}{lccc} x_1(K) \rightarrow & & \overset{\downarrow}{2} & 1 \\ x_2(-K) \rightarrow & 3 & 2 & 1 \\ \hline y(0) = & & 4 + 1 & = 5 \end{array}$$

$\boxed{y(0) = 5}$

$\boxed{n=1}$ $y(1) = \sum_{K=-\infty}^{\infty} x_1(K)\, x_2(1-K)$

$$\begin{array}{lccc} x_1(k) \rightarrow & \overset{\downarrow}{2} & 1 & \\ x_2(1-K) \rightarrow & 3 & 2 & 1 \\ \hline & 6 + 2 & & = 8 \end{array}$$

$\boxed{y(1) = 8}$

$\boxed{n=2}$ $y(2) = \sum_{K=-\infty}^{\infty} x_1(K)\, x_2(2-K)$

$$\begin{array}{lcccc} x_1(K) \rightarrow & \overset{\downarrow}{2} & 1 & & \\ x_2(2-K) \rightarrow & & 3 & 2 & 1 \\ \hline & & 3 & & \end{array}$$

$\boxed{y(2) = 3}$

$n = 3 \quad y(3) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(3-k)$

$x_1(k) \qquad \overset{\downarrow}{2} \quad 1$

$x_2(3-k) \qquad\qquad 3\ 2\ 1$

no overlap $\qquad \boxed{y(3) = 0}$

$n = -1 \quad y(-1) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(-1-k)$

$x_1(k) = \quad \overset{\downarrow}{2} \quad 1$

$x_2(-1-k) = 3\ 2\ \underline{1}$

$2 \qquad \boxed{y(-1) = 2}$

$n = -2 \quad y(-2) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(-2-k)$

$x_1(k) = \qquad \overset{\downarrow}{2} \quad 1$

$x_2(-2-k)\ 3\ 2\ 1$

no overlap

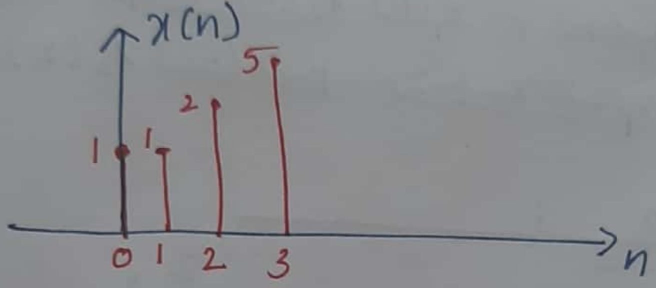
$y(n) = \{2, \underset{\uparrow}{5}, 8, 3, \}.$

③ - A discrete-time LTI system is characterized by the impulse response $h[n] = u[n]$ find its output when the input is

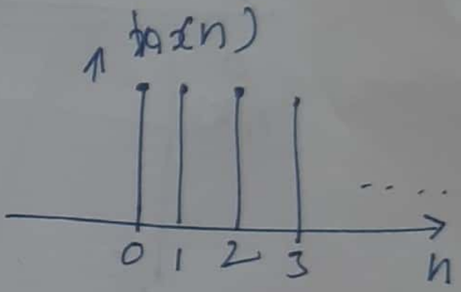
$$x[n] = \{1, 1, 2, 5\}$$

also find $y(-19)$, $y(0)$, $y(2)$, $y(15)$ & $y(28)$

Ⓘ $x[n] = \{1, 1, 2, 5\}$



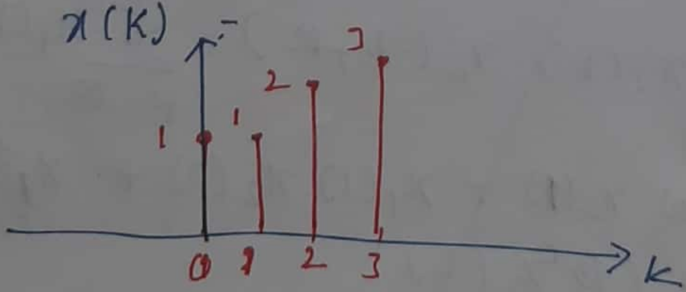
$$h(n) = u(n) = \begin{cases} 1; & n \geq 0 \\ 0; & n < 0 \end{cases}$$



II $y(n) = x(n) * h(n)$

$$\triangleq \sum_{K=-\infty}^{\infty} x(K) . h(n-K) .$$

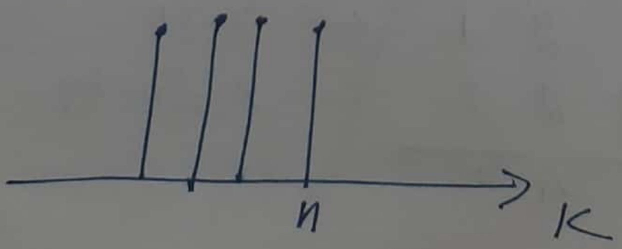
III



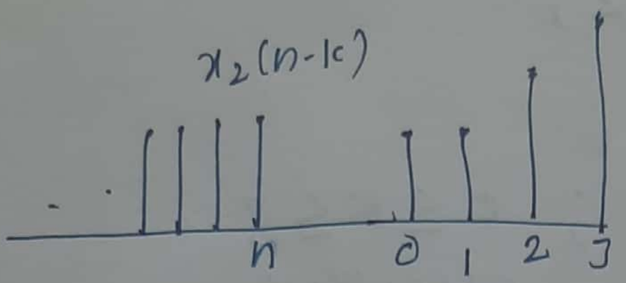
x

$$h(n-K) = \begin{cases} 1, & n-K \geq 0 \text{ (or) } K \leq n \\ 0; & K > n \end{cases}$$

↑h(n-K)



(i) for $n < 0$



$y(n) = 0$

(ii) $n = 0$; $y(0) = 1 \cdot 1 = 1$

$n = 1$ $\quad y(1) = 1 \cdot 1 + 1 \cdot 1 = 2$

$n = 2$ $\quad y(2) = 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 2 = 4$

$n = 3$ $\quad y(3) = 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 3 = 9$

$n = 4$ $\quad y(4) = 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 3 = 9$

$\vdots$

$$y[n] = \{\underset{\uparrow}{1}, 2, 4, 9, 9, 9 \ldots - \}$$

$$\boxed{\begin{array}{lll} y[-19] = 0 & y(2) = 4 & y(28) = 9 \\ y(0) = 1 & y(15) = 9 & \end{array}}$$

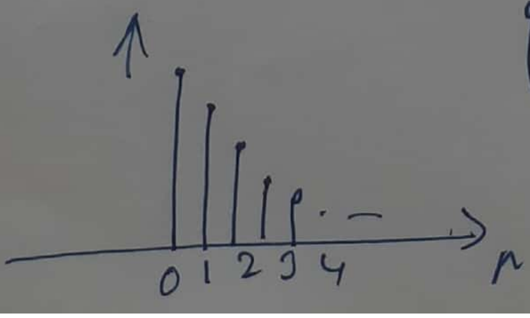
④ $x(n) = n+2$; $0 \le n \le 3$

$h(n) = a^n u(n)$

n	$n+2$
0	2
1	3
2	4
3	5

I $\quad x(n) = \{\underset{\uparrow}{2}, 3, 4, 5\}$

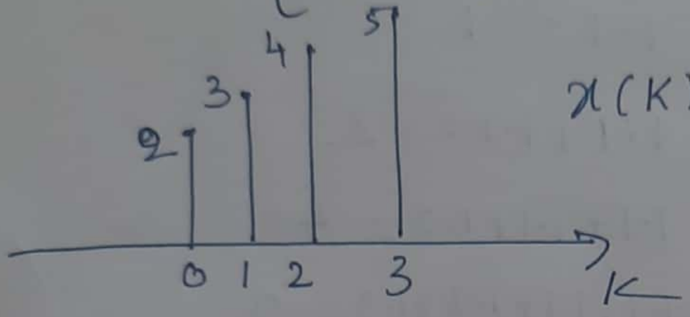
$$h(n) = a^n u(n) = \begin{cases} a^n ; & n \ge 0 \\ 0 ; & n < 0 \end{cases}$$



II $y(n) = x(n) * h(n)$

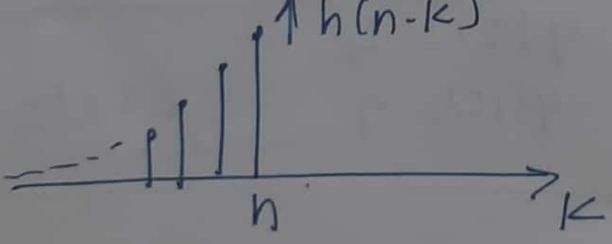
$$\triangleq \sum_{k=-\infty}^{\infty} x(k)\cdot h(n-k)$$

III $x(k) = \begin{cases} k+2 ; & 0 \le k \le 3 \\ 0 ; & \text{elsewhere} \end{cases}$



$x(k) = \{\underset{\uparrow}{2}, 3, 4, 5\}$

$h(n-k) = \begin{cases} a^{n-k} ; & n-k \ge 0 \text{ or } -k \ge -n \\ & \boxed{k \le n} \\ 0 ; & k > n \end{cases}$



IV (i) $n < 0$ $y(n) = \underline{\underline{0}}$

(ii) ?

solve it

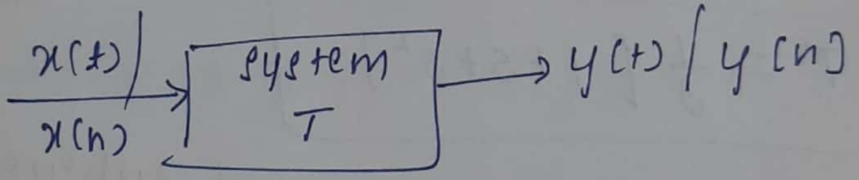
Module - III

LTI system Properties in terms of impulse response /.

system interconnection, Memoryless, causal, stable, invertible & Deconvolution and step response.

① system viewed as interconnection of operations :.

(*) A system is vieweds as interconnection of operations that transforms an i/p into an output signal



$$y(t) = T\{x(t)\}$$

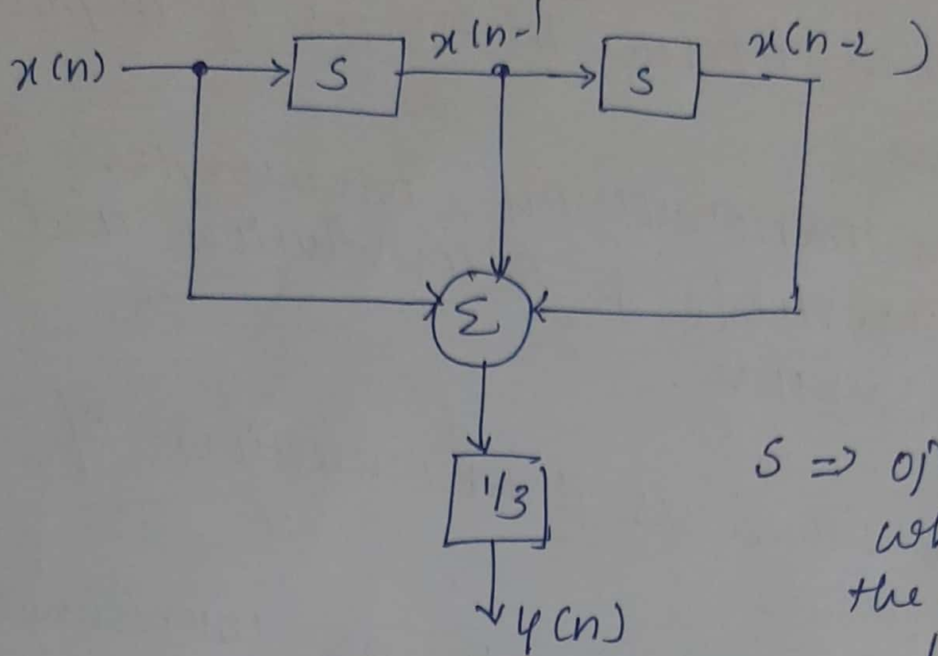
where 'T' ⇒ operator

lly $y[n] = T\{x[n]\}$

① find the overall operator of a system whose output signal y(n) is given by

$$y(n) = \frac{1}{3}[x[n] + x[n-1] + x[n-2]]$$

Also draw the block diagram representation



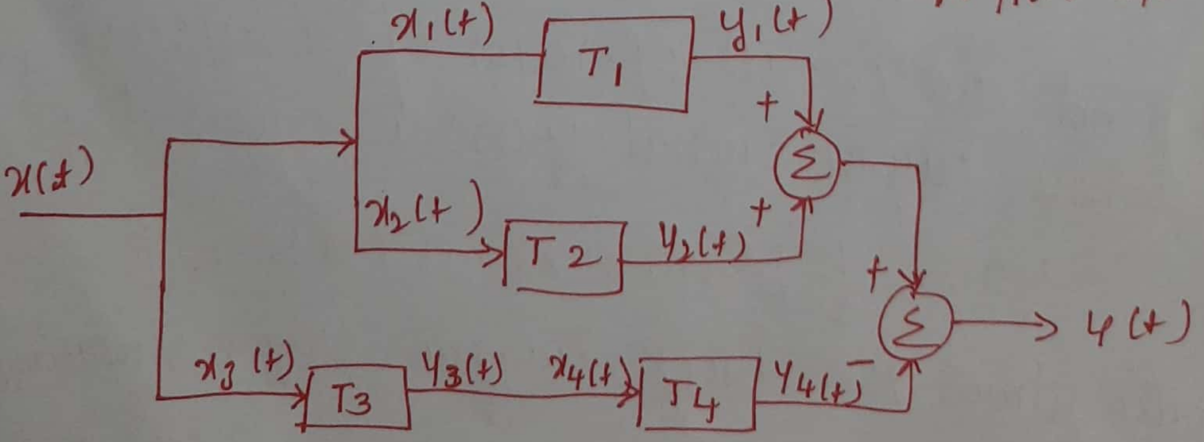
S ⇒ operator which delays the signal by one unit

.˙. $T = \{\ \}x$ is the operation performed by the system of i/p

$$\boxed{T = \frac{1}{3}[1+S+S^2]}$$

② A system consiets of several subsystems connected as shown in fig below. find the operator 'T' relating x(t) to y(t) for the subsystem operators given by

~~$T_1 : y_1(t) = x_1(t)$~~ x



$T_1 : y_1(t) = x_1(t)\, x_1(t-1)$

$T_2 = y_2(t) = |x_2(t)|$

$T_3 = y_3(t) = 1 + 2x_3(t)$

$T_4 = y_4(t) = \cos(x_4(t))$

Soln :- from above fig

$$y(t) = \{y_1(t) + y_2(t)\} - y_4(t)$$

$$= \{x_1(t) \cdot x_1(t-1) + |x_2(t)|\} - \cos(x_4(t))$$

$$= \{x_1(t) \cdot x_1(t-1) + |x_2(t)|\} - \cos(y_3(t))$$

$$= \{x_1(t) \cdot x_1(t-1) + |x_2(t)|\} - \cos(1 + 2x_3(t))$$

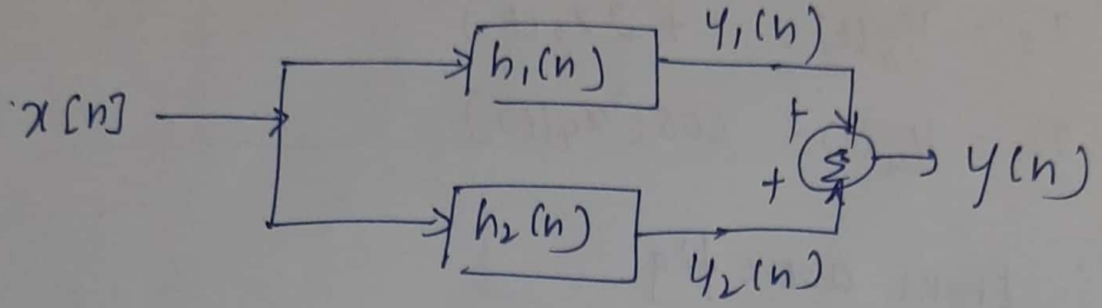
$$x_1(t) = x_2(t) = x_3(t) = x(t)$$

$$\therefore \boxed{y(t) = \{x(t)\, x(t-1) + |x(t)|\} - \cos(1 + 2x(t))}$$

②

Properties of LTI systems

(1) System connected in parallel



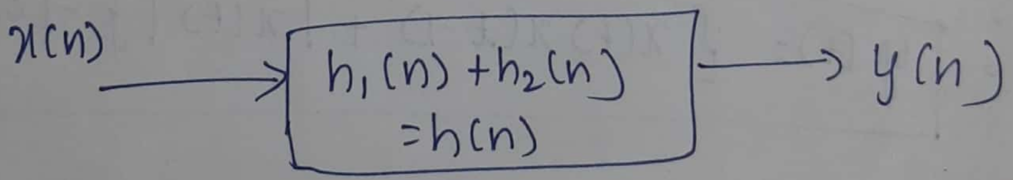
$y[n] = y_1(n) + y_2(n)$

$y(n) = [x(n) * h_1(n)] + [x(n) * h_2(n)]$

using distributive property we get write the above eqn as

$\{[x(n) * [h_1(n) + h_2(n)]]\} = \{[x(n) * h_1(n)] + [x[n] * h_2(n)]\}$

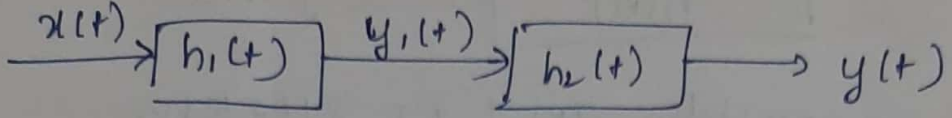
$y(n) = x(n) * [h_1(n) + h_2(n)]$



$$\boxed{h(n) = h_1(n) + h_2(n)}$$

lly $\boxed{h(t) = h_1(t) + h_2(t)}$

(2) System connected in series / cascade



$$y(t) = y_1(t) * h_2(t)$$

$$= [x(t) * h_1(t)] * h_2(t)$$

using associative property we can rewrite the above eqn as

ie $x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$

$$\therefore\ y(t) = x(t) * [h_1(t) * h_2(t)]$$

$$= x(t) * h(t)$$

$$\boxed{h(t) = h_1(t) * h_2(t)}$$

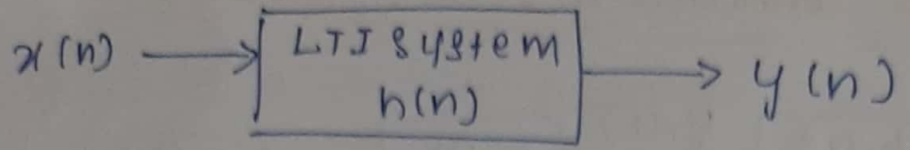
||ly $\boxed{h(n) = h_1(n) * h_2(n)}$

(3) memoryless system

A system is said to be memoryless if its output at any time depends only on the present value of the input

A discrete time LTI system is said to be memoryless if and only if its impulse response is of the form

$\boxed{h(n) = C\cdot\delta(n)}$ where C = constant



Let us consider an LTI system represented by its impulse response h(n)

$y(n) = x(n) * h(n)$

if $h(n) = c\,\delta(n)$

then $y(n) = x(n) * c\,\delta(n)$

$\boxed{y(n) = c\,\cancel{x}\delta(n)}$ $\boxed{y(n) = c\,x(n)}$

The above eqn means that nth value of the o/p y(n) depends only on the nth value of i/p x(n)
[condition for a system to have no memory]

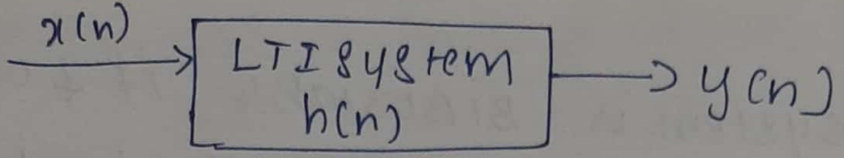
IIly for continuous time signal

$\boxed{h(t) = c\,\delta(t)}$ c = arbitraly constant

(4) Causality

A ~~discrete~~ LTI system is causal if & only if its impulse response h(n) is zero for n<0. i.e causal seqn of h(n) automatically implies a causal system

$\boxed{h(n) = 0}$; n < 0



$$y(n) = x(n) * h(n)$$

$$\triangleq \sum_{K=-\infty}^{\infty} x(K) \cdot h(n-K) \quad — ①$$

if $h(n) = 0 ; \; n < 0$

then $h(n-K) = 0 ; \; n-K < 0$ or $K > n$.

using this in the above eqn① we get

$$y(n) \triangleq \sum_{K=-\infty}^{n} x(K) \cdot h(n-K) \longrightarrow ②$$

from eq ② it is clear that output y(n) at any time 'n' is weighted sum of the value of input x(K) for K less than or equal to 'n' i.e., only present and past inputs. ∴ the system is causal

lly for continuous time signal

$$\boxed{h(t) = 0 ; \; t < 0}$$

Ex $n = 5$

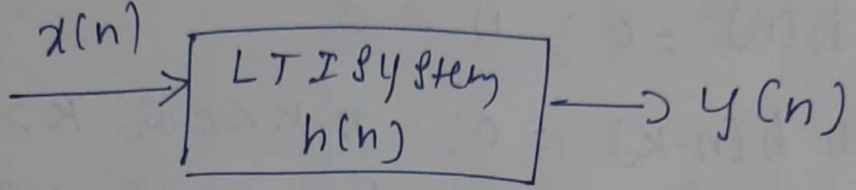
$$y(5) = \sum_{K=-\infty}^{5} x(K) \cdot h(n-K)$$

⑥ Stability

A LTI system is BIBOstable if & only if its impulse response h(n) is absolutely summable

i.e. $\boxed{\sum_{k=-\infty}^{\infty} |h(k)| < \infty}$ for BIBO Stability

- proof



$$y(n) = x(n) * h(n)$$
$$= h(n) * x(n)$$
$$\triangleq \sum_{k=-\infty}^{\infty} h(k) \cdot x(n-k) \longrightarrow ①$$

Taking ~~maginute~~ magnitudes on both sides of eq ①

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} h(k) \cdot x(n-k) \right| \longrightarrow ②$$

using triangular inequality, that the sum of the magnitudes is $\leq$ to the magnitudes of the sum, we get

$$|y(n)| \leq \sum_{k=-\infty}^{\infty} |h(k) \cdot x(n-k)| \longrightarrow ③$$

since $x(n)$ is bounded we have

$$|x(n)| \leq M_x < \infty$$

$$\therefore |x(n-k)| \leq M < \infty \quad \text{— ④}$$

using this in eq ③

$$|y(n)| \leq \sum_{k=-\infty}^{\infty} |h(k)| \cdot M$$

$\therefore$ if $\sum_{k=-\infty}^{\infty} |h(k)|$ is finite then

$|y(n)| \leq M_y < \infty$ bounded o/p

$\therefore$ for BIBO stable, $\sum_{k=-\infty}^{\infty} |h(k)$ should be absolutely summable

|||ly for continuous time signal

$$\int_{\tau=-\infty}^{\infty} h(\tau)\, d\tau < \infty$$

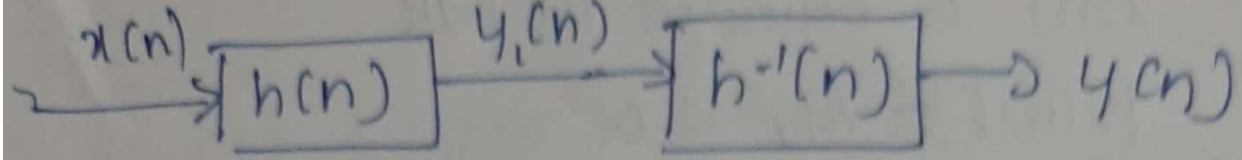
⑦ Invertiblity

A discrete LTI system with impulse response $h(n)$ is said to be invertible if & only if

$$\boxed{h(n) * h^{-1}(n) = \delta(n)}$$

& $h^{-1}(n)$ = Impulse response of an inverse system

⑧



$y(n) = y_1(n) * h^{-1}(n) = y_1(n) * h^{-1}(n)$

$= [x(n) * h(n)] * h^{-1}(n)$

using associative property

$= x(n) * [h(n) * h^{-1}(n)]$

$=$ If $h(n) * h^{-1}(n) = \delta(n)$

then

$y(n) = x(n) * \delta(n)$

$= x(n)$

||ly ~~get~~ for continuous time LTI system

$$\boxed{h(t) * h^{-1}(t) = \delta(t)}$$

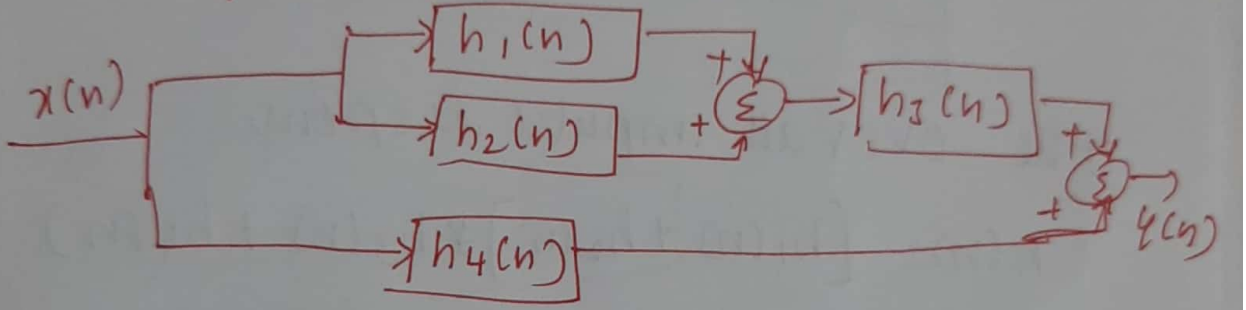
① consider the interconnection of LTI system shown in fig below, The impulse response of each system is given as

$h_1(n) = u(n)$

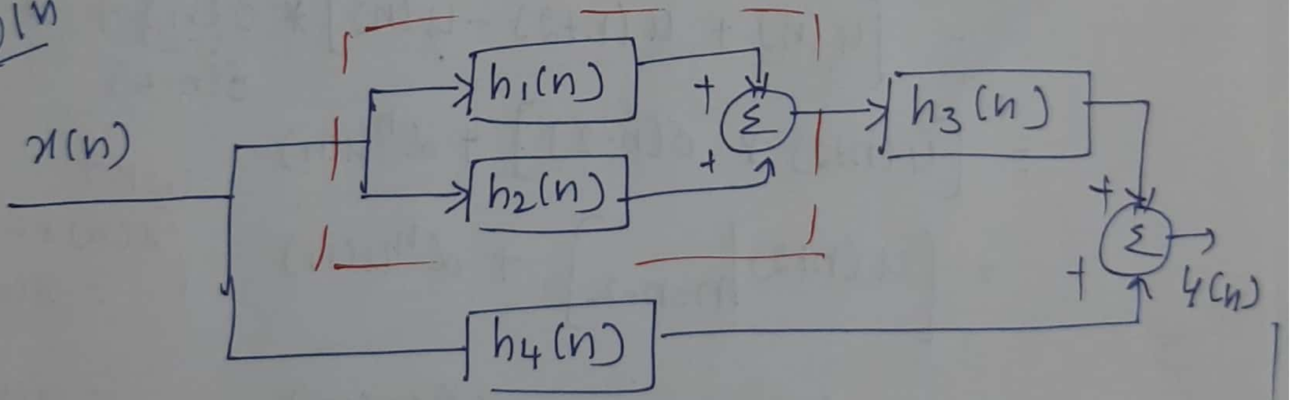
$h_2(n) = u(n+2) - u(n)$

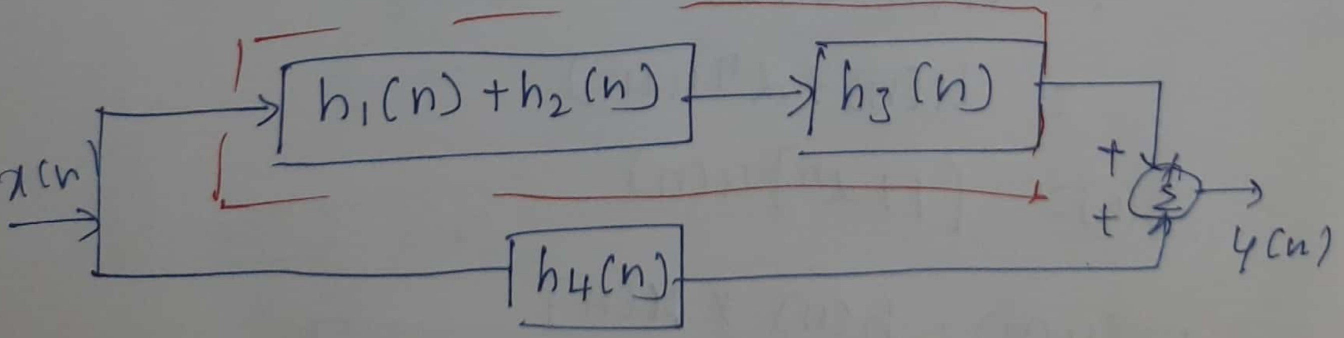
$h_3(n) = \delta(n-2)$ & $h_4(n) = \alpha^n u(n)$

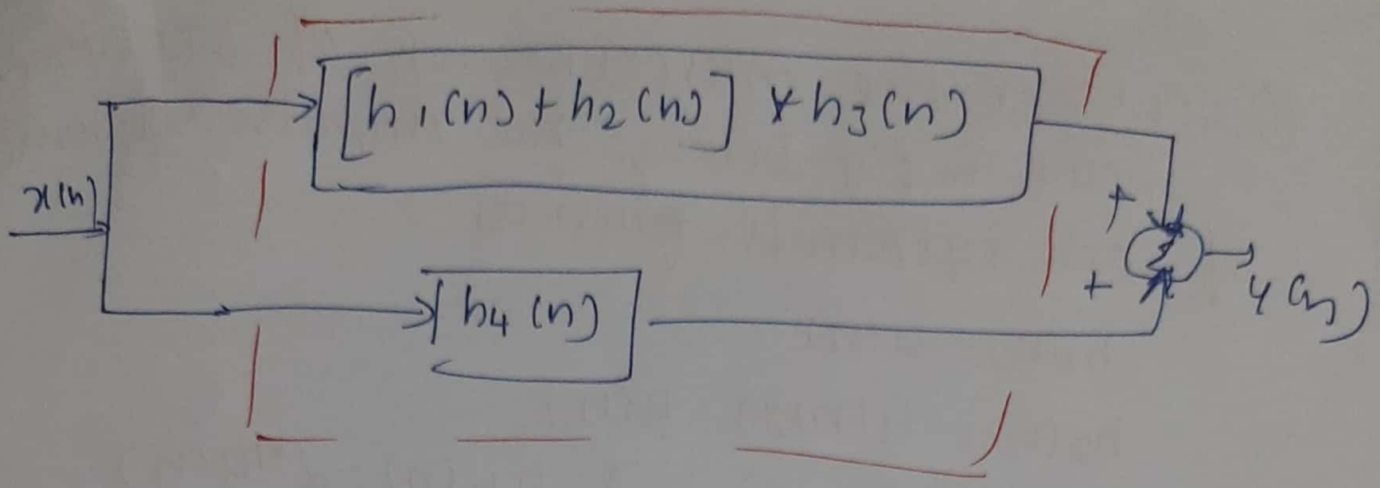
find the overall impulse response of the system

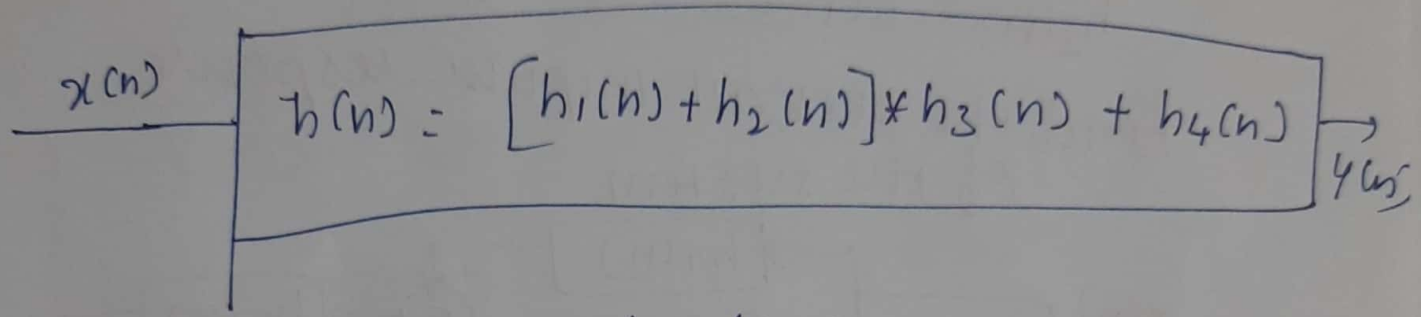


soln'









The overall impulse response

$$h(n) = [h_1(n) + h_2(n)] * h_3(n) + h_4(n)$$

$$= [\cancel{u(n)} + u(n+2) - \cancel{u(n)}] * \delta(\underset{-2}{n}) + \alpha^n u(n)$$

$$\delta(n-2)$$

$$= [u(n+2) * \delta(n-2)] + \alpha^n u(n)$$

$$= \left[u(n+2)\big|_{n=n-2}\right] + \alpha^n u(n)$$

WKT
$[x(n) * \delta(n-k)$
$= x(n)|_{n=n-k}$
$= x(n-k)]$

$$= u(n-2+2) + \alpha^n u(n)$$

$$= u(n) + \alpha^n u(n)$$

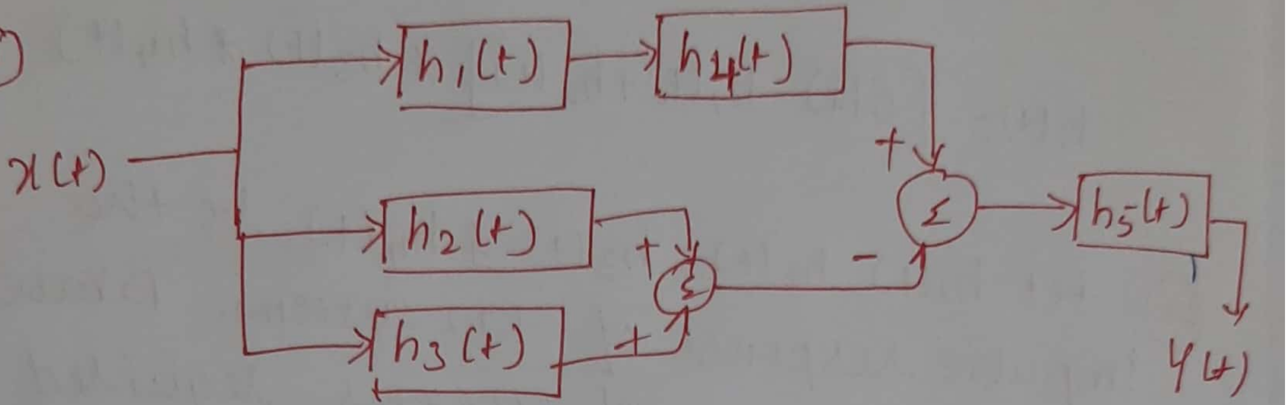
$$h(n) = [1+\alpha^n]\, u(n)$$

$$\therefore\ y(n) = h(n) * x(n)$$

$$\boxed{y(n) = [1+\alpha^n]\, u(n) * x(n)}$$

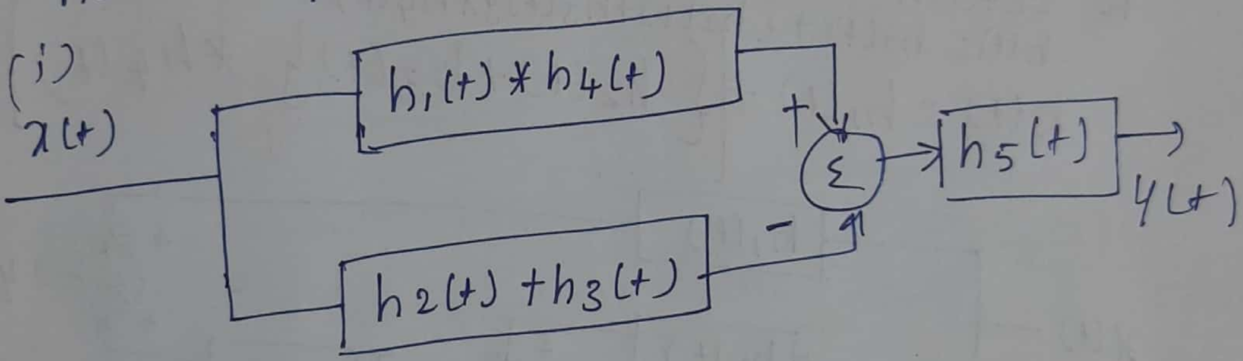
② Find the overall impulse response h(t) interms of impulse response of each subsystem for the system shown below

①

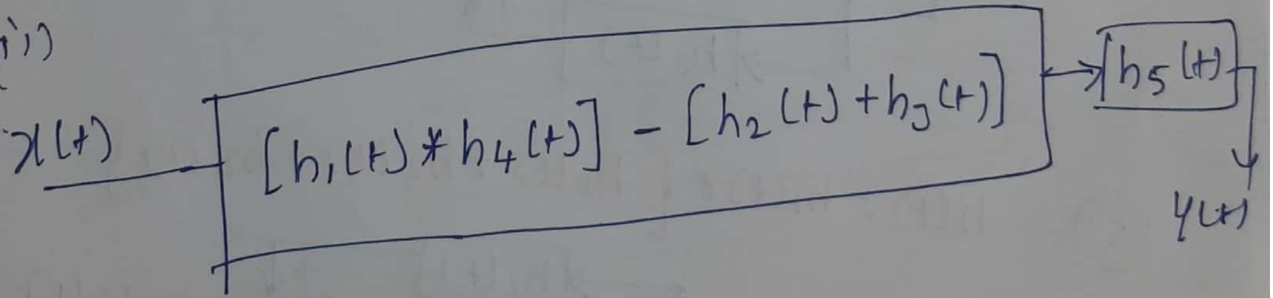


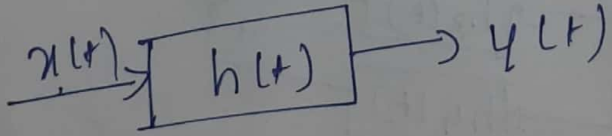
$h(t) = h_1(t)$

(i)



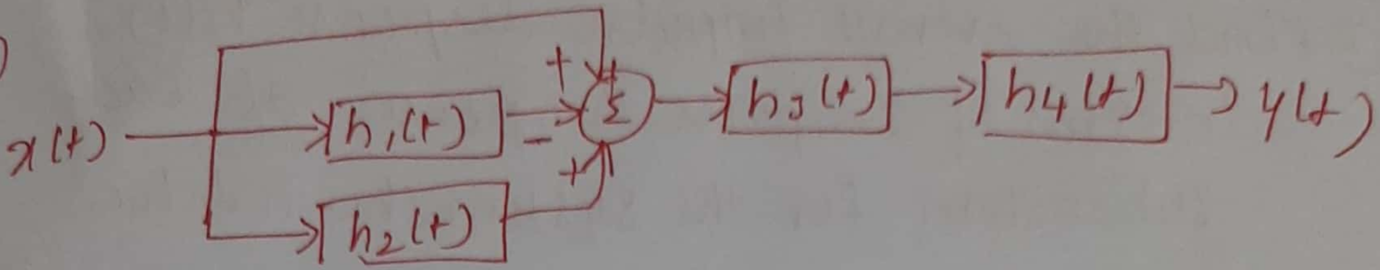
(ii)





$$h(t) = \left\{ [h_1(t) * h_4(t)] - [h_2(t) + h_3(t)] \right\} * h_5(t)$$

③

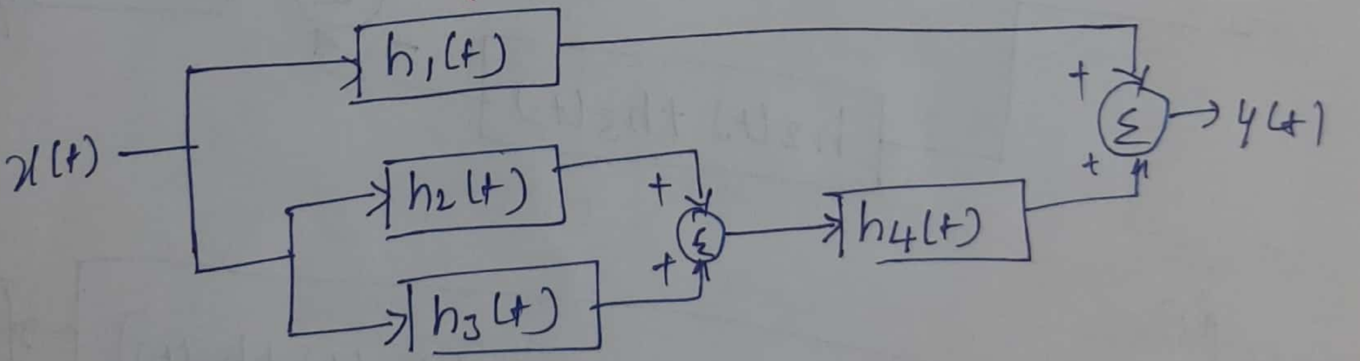


$$h(t) = [\delta(t) - h_1(t) + h_2(t)] * h_3(t) * h_4(t)$$

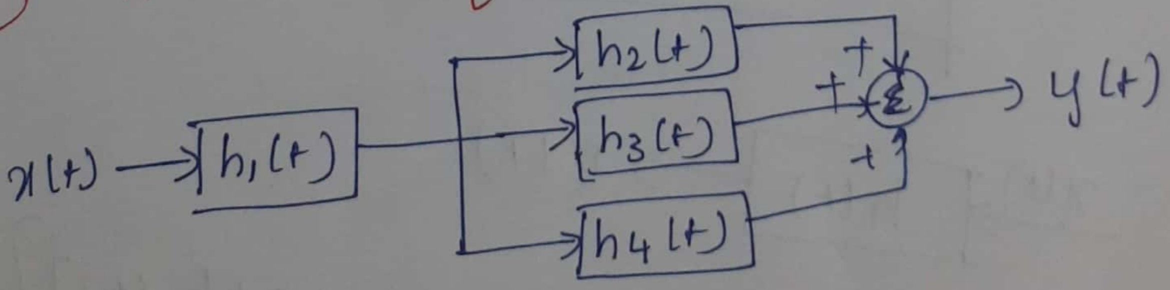
④ Let $h_1(t)$, $h_2(t)$, $h_3(t)$ & $h_4(t)$ be the impulse response of LTI systems. Draw the interconnection of systems required to obtain the overall impulse response

$$h(t) = h_1(t) + \{h_2(t) + h_3(t)\} * h_4(t)$$

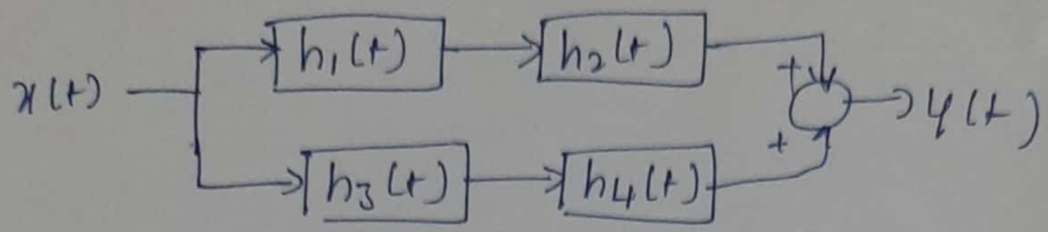
$$h(t) = h_1(t) + \left[\{h_2(t) + h_3(t)\} * h_4(t)\right]$$



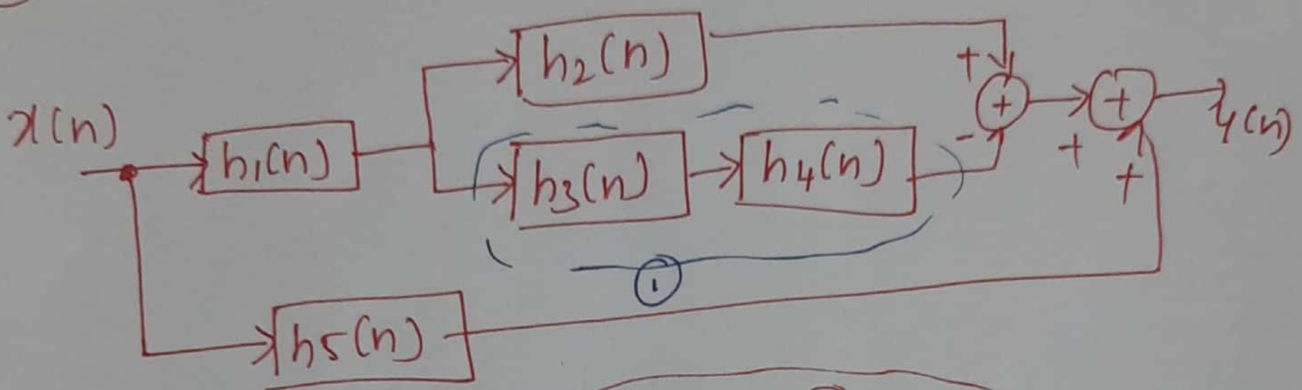
⑤ $h(t) = h_1(t) * \{h_2(t) + h_3(t) + h_4(t)\}$

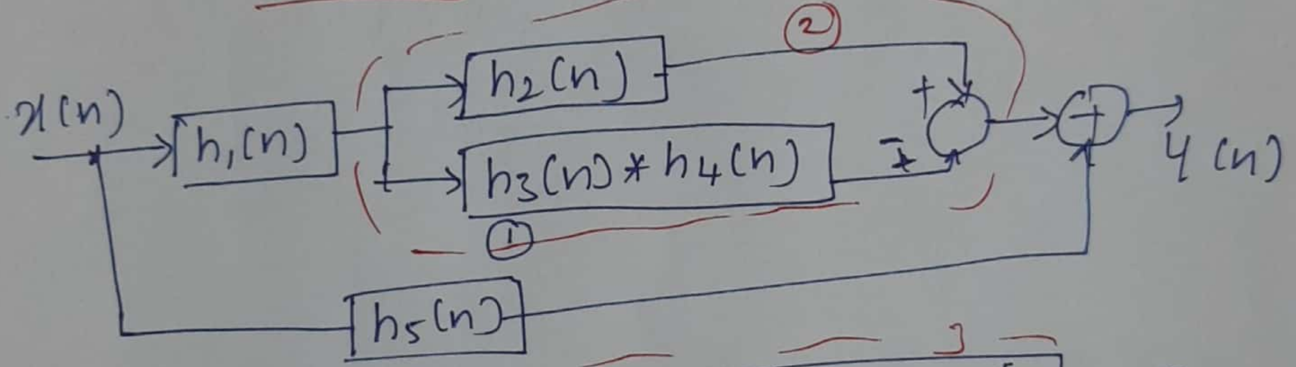


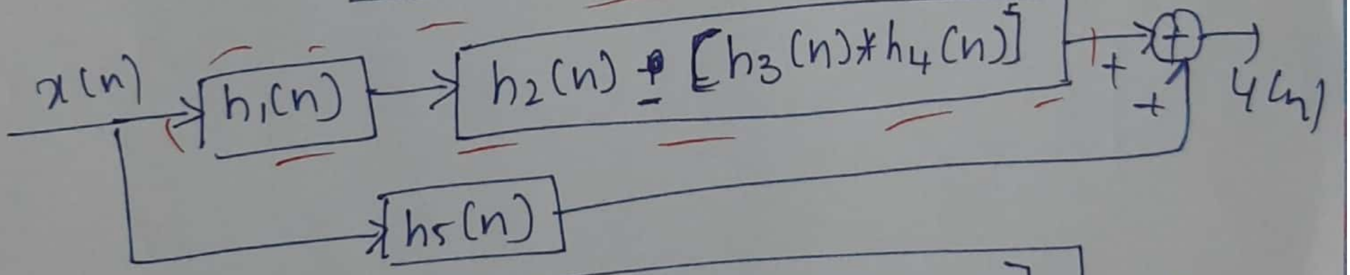
⑥ $h(t) = h_1(t) * h_2(t) + h_3(t) * h_4(t)$

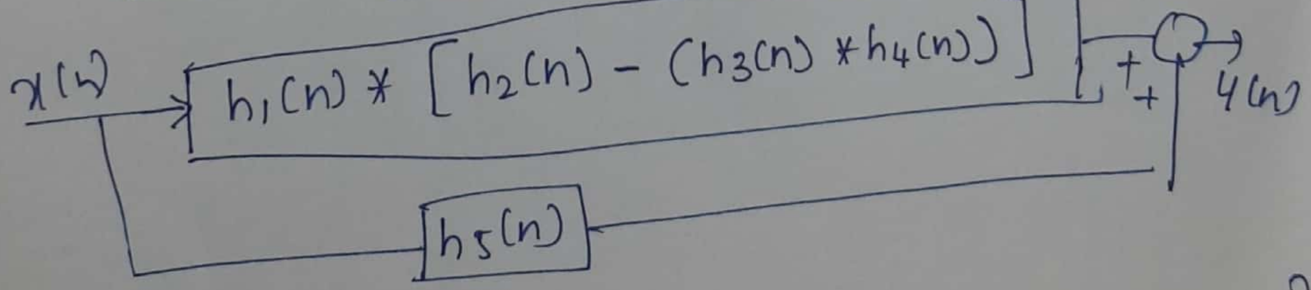


⑦ Express h(n)









$$h(n) = h_5(n) + \left\{ h_1(n) * \left[h_2(n) - (h_3(n) * h_4(n)) \right] \right\}$$

⑧

Problems on properties of LTI system

Summary:

Property	Condition to prove property	
	Discrete time signals	Continuous time signals
1. Memory Less	$h(n) = c.\delta(n)$	$h(t) = c.\delta(t)$
2. Causal	$h(n)=0$ for $n<0$	$h(t)=0$ for $t<0$
3. Stable	$\sum_{K=-\infty}^{\infty} \lvert h(k) \rvert < \infty$	$\int_{-\infty}^{\infty} \lvert h(\tau) \rvert d\tau < \infty$
4. Invertibility	$h(n) * h^{-1}(n) = \delta(n)$	$h(t) * h^{-1}(t) = \delta(t)$

Determine whether the following systems for the given impulse response are (i) memoryless (ii) causal (iii) stable Justify your answers

(1) $h(t) = 3\delta(t)$ $\quad h(t) = \begin{cases} 3 ; & t=0 \\ 0 ; & t \neq 0 \end{cases}$

(i) since $h(t)$ is of the form $c.\delta(t)$

$\therefore$ it is memoryless

(ii) since $h(t)=0$ for $t<0$

$\therefore$ causal

(iii) $$\int_{-\infty}^{\infty} \lvert h(\tau) \rvert d\tau = \int_{-\infty}^{\infty} 3\delta(\tau) d\tau = \int_{\tau=}$$

$$= 3.\delta(0)\, d\tau = \underline{\underline{3}} < \infty$$

$\therefore$ Stable

(2) $h(t) = e^{-3t}\, u(t-1)$

$$h(t) = \begin{cases} e^{-3t} & ; \; t > 1 \\ 0 & ; \; t < 1 \end{cases}$$

(i) since $h(t) \neq c.\delta(t)$ $\therefore$ not memoryless

(ii) $h(t) = 0$ for $t < 0$ $\therefore$ causal

(iii) $$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau = \int_{1}^{\infty} e^{-3\tau}\, d\tau = \left.\frac{e^{-3\tau}}{-3}\right]_{1}^{\infty}$$

$$= \frac{e^{-3\infty} - e^{-3}}{-3} = \frac{0 - e^{-3}}{3} = \frac{e^{-3}}{3} < \infty$$

$\therefore$ Stable

$$\left[\begin{array}{l} e^{\infty} = 1 + \ldots + \infty = \infty \\ e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0 \end{array}\right.$$

(3) $h(t) = e^{2t}\, u(t-1)$

$$h(t) = \begin{cases} e^{2t} & ; \; t > 1 \\ 0 & ; \; t < 0 \end{cases}$$

(i) since $h(t) \neq c\delta(t)$ $\therefore$ NML

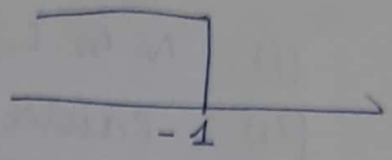
(ii) $h(t) = 0$ for $t < 0$ $\therefore$ causal

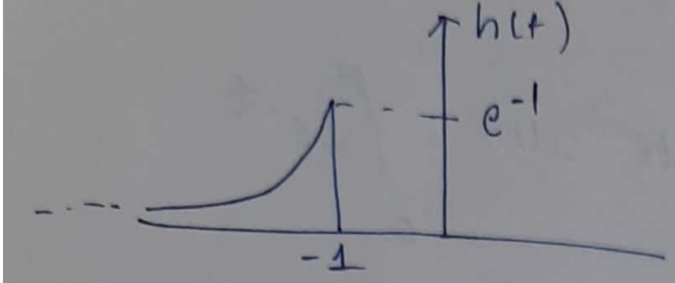
(iii) $$\int_{-\infty}^{\infty} |h(\cancel{t})|\, dt = \int_{1}^{\infty} e^{2t}\, dt = \left.\frac{e^{2\tau}}{2}\right]_{1}^{\infty} = \infty$$

$\therefore$ unstable

(4) $h(t) = e^{t} u(-1-t)$

$u(-1-t) = u(-t-1)$





(i) $h(t) \neq c \cdot \delta(t)$ ∴ NML

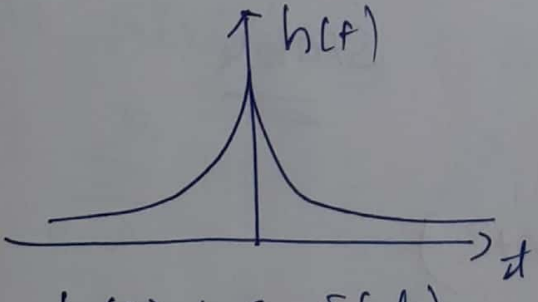
(ii) $h(t) \neq 0 \quad t < 0$ ∴ Non-causal

(iii) $$\int_{-\infty}^{\infty} |h(t)| \, d\tau = \int_{-\infty}^{-1} e^{t} \, d\tau = e^{\tau}\Big]_{-\infty}^{-1}$$

$$= e^{-1} < \infty$$ ∴ Stable

⑤ $h(t) = e^{-4|t|}$

$h(t) = e^{4t} u(-t) + e^{-4t} u(t)$



(i) $h(t) \neq c \cdot \delta(t)$ ∴ NML

(ii) $h(t) \neq 0 \; ; t < 0$ ∴ Non-causal

(iii) $$\int_{-\infty}^{\infty} |h(\tau)| \, d\tau = \int_{-\infty}^{0} e^{4\tau} \, d\tau + \int_{0}^{\infty} e^{-4\tau} \, d\tau$$

$$= \frac{e^{4\tau}}{\tau}\Big]_{-\infty}^{0} + \frac{e^{-4\tau}}{-4}\Big]_{0}^{\infty} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} < \infty$$

∴ Stable

⑥ $h(t) = t e^{-t} u(t)$

(i) N.M.L

(ii) causal

(iii) $\int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{\infty} t e^{-t} u(t) = \int_{0}^{\infty} t e^{-t}$

$= 1 < \infty$ stable

⑦ $h(n) = \left(\frac{1}{2}\right)^n u(n)$

$h(n) = \begin{cases} (1/2)^n & ; n \geq 0 \\ 0 & ; n < 0 \end{cases}$

(i) $h(n) \neq C \cdot \delta(n)$ $\therefore$ NML

(ii) $h(n) = 0$; $n < 0$ $\therefore$ causal

(iii) $\sum_{K=-\infty}^{\infty} |h(K)| = \sum_{K=0}^{\infty} (1/2)^K = \frac{1}{1 - 1/2} = 2 < \infty$

WKT $\sum_{K=0}^{\infty} a^k = \frac{1}{1-a}$

Stable

⑧ $h(n) = (0.99)^n u(n+3)$

(i) $h(n) = \begin{cases} (0.99)^n & ; n \geq -3 \\ 0 & ; n < -3 \end{cases}$

(i) $h(n) \neq c\delta(n)$ $\therefore$ NML

(ii) $h(n) \neq 0$ $n < 0$, $\therefore$ Non causal

since $n = -3, -2, -1$ $h(n)$ has value

(iii) $\sum_{K=-\infty}^{\infty} |h(K)| = \sum_{K=-3}^{\infty} |(0.99)^K|$

$$\boxed{\text{wkt } \sum_{n=K}^{\infty} a^n = \frac{a^K}{1-a} \quad |a|<1}$$

$$= \frac{(0.99)^{-3}}{1-0.99} = 103.06 < \infty$$

∴ Stable

⑨ $h(n) :- n(1/2)^n u(n)$

$$h(n) = \begin{cases} n(1/2)^n ; & n \geq 0 \\ 0 ; & n < 0 \end{cases}$$

(ii) since $h(n) \neq c.\delta(n)$ ∴ NML

(ii) $h(n) = 0 ; n < 0$ ∴ Causal

(iii) $\sum_{K=\infty}^{\infty} |h(K)| = \sum_{K=0}^{\infty} K(1/2)^K$

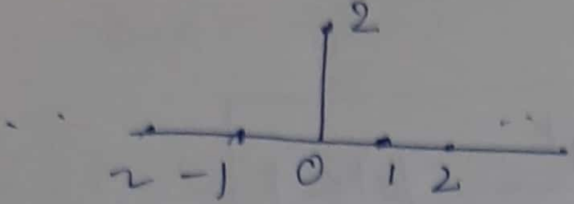
$$\boxed{\sum_{n=0}^{\infty} n a^n = \frac{a}{(1-a)^2} \quad ; |a|<1}$$

$$= \frac{1/2}{(1-1/2)^2} = 2 < \infty$$

∴ Stable

⑩ (i)

⑩ $h(n) = 2u(n) - 2u(n-1)$



$h(n) = 2 \; ; n = 0$

$\quad\quad 0 \; ; n \neq 0$

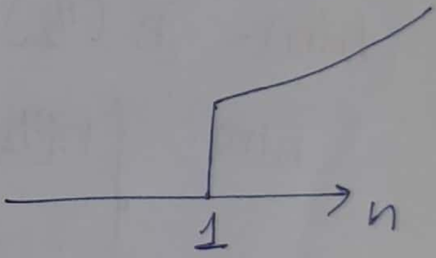
$h(n) = 2\delta(n)$

(i) since $h(n) = c.\delta(n)$ ∴ Memoryless

(ii) Since $h(n) = 0 \; ; n < 0$ Causal

(iii) $\sum_{K=\infty}^{\infty} |h(K)| = h(0) = 2 < \infty$ ∴ Stable

⑪ $h(n) = e^{2n}u(n-1)$



(i) $h(n) \neq c.\delta(n)$

∴ system is $\boxed{NML}$

(ii) $h(n) = 0 \; ; n < 0$ ∴ system is causal

(iii) $\sum_{K=-\infty}^{\infty} |h(K)| = \sum_{K=1}^{\infty} e^{2K} = \sum_{K=1}^{\infty} (e^2)^K$

$$\boxed{\sum_{n=1}^{\infty} \alpha^n = \frac{\alpha}{1-\alpha} \text{ iff } |\alpha| < 1}$$

$= \frac{e^2}{1-e^2}$

$|e^2| \not< 1$ ∴ $= \infty$

∴ unstable

⑫ $h(n) = (0.5)^{|n|}$

$$h(n) = (0.5)^{-n} u(-n-1) + (0.5)^{n} u(n)$$

(i) $h(n) \neq c.\delta(n)$ ∴ NML

(ii) $h(n) \neq 0 \, ; \, n<0$ ∴ causal

(iii) $$\sum_{k=-\infty}^{\infty} |h(k)|$$

$$= \sum_{k=-\infty}^{-1} (0.5)^{-k} + \sum_{k=0}^{\infty} (0.5)^{k}$$

Substitute $k = -m$ in 1st summation only

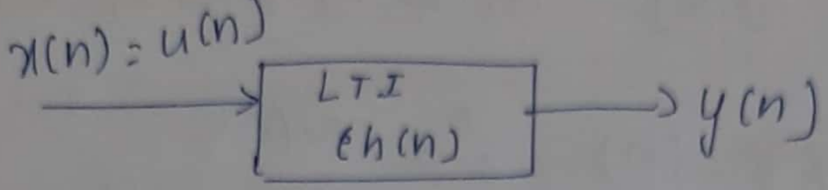
$$= \sum_{m=\infty}^{1} (0.5)^{m} + \sum_{k=0}^{\infty} (0.5)^{k}$$

$$= \sum_{m=1}^{\infty} (0.5)^{m} + \sum_{k=0}^{\infty} (0.5)^{K}$$

$$= \frac{0.5}{1-0.5} + \frac{1}{1-0.5}$$

$$= 1 + 2 = 3 < \infty$$ ∴ stable

Step Response of a LTI system



Let us consider a LTI system as shown in fig above

* if i/p of the system is a step fun then the response of the LIT system is step response

$$s(n) = u(n) * h(n)$$

$$= h(n) * u(n) \quad \text{[commutative property]}$$

$$= \sum_{K=-\infty}^{\infty} h(K) \cdot u(n-K)$$

define $u(n-K) \Rightarrow$

$$u(n) = \begin{cases} 1 ; & n \geq 0 \\ 0 ; & n < 0 \end{cases}$$

$$\therefore u(n-K) = \begin{cases} 1 ; & n-K \geq 0 \text{ (or) } K \leq n \\ 0 ; & K > n \end{cases}$$

$$\therefore \boxed{s(n) = \sum_{K=-\infty}^{n} h(K)}$$

running sum of impulse response

IIIly

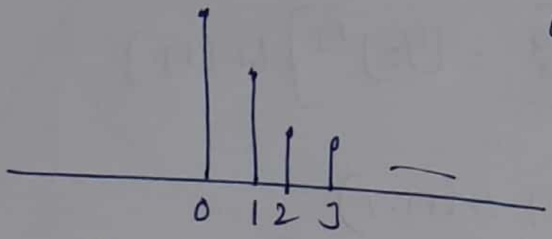
$$\boxed{h(t) = \int_{-\infty}^{t} h(\tau) d\tau}$$

running integral of impulse respon

Find the step response of an LTI system represented by the foll impulse response

① $h(n) = (1/2)^n u(n) = (1/2)^n \; ; n \ge 0$

$0 \; ; n < 0$



$$s(n) \triangleq \sum_{k=-\infty}^{n} h(k)$$

(i) for $n < 0$ $\boxed{h(k) = 0}$

(ii) for $n \ge 0$

$$s(n) = \sum_{k=0}^{n} (1/2)^k = \frac{1-(1/2)^{n+1}}{1-1/2}$$

$$\boxed{\text{wkt} \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a} \quad |a| \neq 1}$$

$$= \frac{1-(1/2)^n.(1/2)}{\frac{2-1}{2}}$$

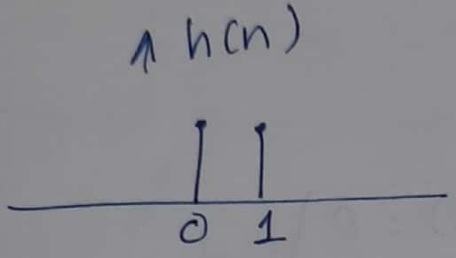
$$= \frac{1-(1/2)^n(1/2)}{1/2} = 2\left[1-(1/2)^n\left(\tfrac{1}{2}\right)\right]$$

$$= 2 - \left(\tfrac{1}{2}\right)^n \times 2 \times \tfrac{1}{2} = \underline{\underline{2-(1/2)^n}}$$

$\therefore$ $s(n) = \begin{cases} 0 ; & n < 0 \\ 2 - (1/2)^n ; & n \geq 0 \end{cases}$

$s(n) = \left[2 - (1/2)^n\right] u(n)$

② $h(n) = \delta(n) + \delta(n-1)$



$$s(n) = \sum_{K=-\infty}^{n} h(K)$$

(i) $n < 0$; $s(n) = 0$

(ii) $n = 0$ $s(0) = 1$

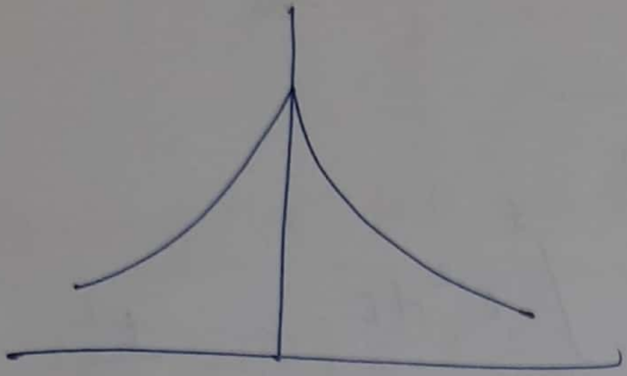
$n \geq 1$ $s(n) = \sum_{K=0}^{n} h(k)$

$= 1 + 1 = 2$

$\therefore$ $s(n) = \begin{cases} 0 ; & n < 0 \\ 1 ; & n = 0 \\ 2 ; & n \geq 1 \end{cases}$

③ $h(t) = e^{-|t)}$

$$h(t) = e^{-t}u(t) + e^{t}u(-t)$$



$$s(t) = \int_{-\infty}^{t} h(\tau)\,d\tau$$

(i) $t < 0$ $\quad s(t) = \int_{-\infty}^{t} e^{\tau}\,d\tau = \underline{\underline{e^{t}}}$

(ii) $t \geq 0$

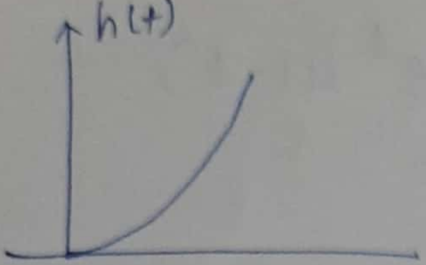
$$s(t) = \int_{-\infty}^{0} e^{+\tau}\,d\tau + \int_{0}^{t} e^{-\tau}\,d\tau$$

$$e^{t}\Big]_{-\infty}^{0} + \frac{e^{-\tau}}{-1}\Big]_{0}^{t} = \underline{\underline{2 - e^{-t}}}$$

$$\boxed{\begin{aligned} &[e^{0} - e^{-\infty}] \\ &\quad + [-e^{t} + e^{0}] \\ &1 - 0 - e^{-t} - 1 \\ &= 2 - e^{-t} \end{aligned}}$$

$$s(t) = \begin{cases} e^{t} & ; t < 0 \\ 2 - e^{-t} & ; t \geq 0 \end{cases}$$

④ $h(t) = t^2 u(t)$



$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$

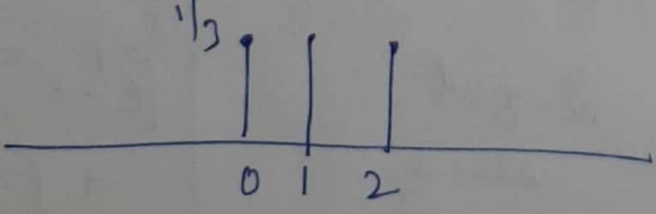
(i) $t < 0 \quad s(t) = 0$

(ii) $t > 0 \quad s(t) = \int_{0}^{t} \tau^2 d\tau = \frac{t^3}{3}$

$$s(t) = \begin{cases} 0 ; & t < 0 \\ \frac{t^3}{3} ; & t > 0 \end{cases}$$

⑤ $h(n) = \frac{1}{3} \sum_{k=0}^{2} \delta(n-k)$

$h(n) = \frac{1}{3} \left[\delta(n) + \delta(n-1) + \delta(n-2) \right]$



$s(n) \triangleq \sum_{k=-\infty}^{n} h(k)$

(i) $n < 0 \quad s(n) = 0$

(ii) $n = 0 \quad s(0) = 1/3$

$n = 1 \quad s(1) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

$n \geq 2$

$s(n) = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$

∴

$$s(n) = \begin{cases} 0 ; & n < 0 \\ 1/3 ; & n = 0 \\ 2/3 ; & n = 1 \\ 1 ; & n \geq 2 \end{cases}$$

Find the impulse response of the ~~fitte~~ system whose step response is given

<u>Note:</u> WKT the step response of the system is given by

$$s(n) = \sum_{K=-\infty}^{n} h(K) = \sum_{K=-\infty}^{n} h(K)$$

expanding this

$$s(n) = h(n) + h(n-1) + h(n-2) + \cdots + h(-\infty) \quad ①$$

$$s(n-1) = \sum_{K=-\infty}^{n-1} h(K)$$

$$s(n-1) = h(n-1) + h(n-2) + \cdots + h(-\infty)$$

<u>to find h(n)</u>

$$s(n) - s(n-1) = h(n)$$

$\therefore$ impulse response of system

$$\boxed{h(n) = s(n) - s(n-1)}$$

for continuous system

$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$

$$\boxed{h(t) = \frac{d\, s(t)}{dt}}$$

$s(n) = \left(\frac{1}{2}\right)^n u(n-1) + 2^n u(-n)$

Sol^n:

$$s(n) = \left(\frac{1}{2}\right)^n u(n-1) + 2^n u(-n)$$

$$s(n-1) = \left(\frac{1}{2}\right)^{n-1} u(n-1-1) + 2^{n-1} u(-n+1)$$

$$s(n-1) = \left(\frac{1}{2}\right)^{n-1} u(n-2) + 2^{n-1} u(-n+1)$$

$$h(n) = s(n) - s(n-1)$$

$$= \left[\left(\frac{1}{2}\right)^n u(n-1) + 2^n u(-n)\right] - \left[\left(\frac{1}{2}\right)^{n-1} u(n-2) + 2^{n-1} u(-n+1)\right]$$

$$\boxed{h(n) = \left(\frac{1}{2}\right)^n u(n-1) + 2^n u(-n) - \left(\frac{1}{2}\right)^{n-1} u(n-2) - \cancel{+}\, 2^{n-1} u(-n+1)}$$

① $\left(\frac{1}{2}\right)^n u(n-1) = \begin{cases} \underline{(1/2)^n} \; ; & n-1 \geq 0 \text{ (or) } \boxed{n \geq 1} \\ 0 \; ; & n < 1 \end{cases}$

② $2^n u(-n) = \begin{cases} \underline{2^n} \; ; & \boxed{n \leq 0} \\ 0 \; ; & n > 0 \end{cases}$

④ $2^{n-1} u(-n+1) = \begin{cases} \underline{\underline{2^{n-1}}} \; ; & -n+1 \geq 0 \text{ (or) } -n \geq -1 \;\; \boxed{n \leq 1} \\ 0 \; ; & n > 1 \end{cases}$

③ $(1/2)^{n-1} u(n-2) = \begin{cases} \underline{\underline{(1/2)^{n-1}}} \; ; & n-2 \geq 0 \;\; \boxed{n \geq 2} \\ 0 \; ; & n < 2 \end{cases}$

case(i)

(i) $\boxed{n=0}$ 1st & 3rd

$$h(n) = (1/2)^n \cancelto{0}{u}(n-1) + 2^n u(-n) - (1/2)^{n-1} \cancelto{0}{u}(n-2) - 2^{n-1} u(-n+1)$$

$$h(n) = 2^n - 2^{n-1}$$

$$h(0) = 2^0 - 2^{0-1} = 1 - 2^{-1} = 1 - \frac{1}{2} = \frac{1}{2} = 0.5$$

$$\boxed{h(0) = 0.5}$$

(ii) $\boxed{n=1}$ ~~2nd~~ 1st & 4th

$$h(n) = (1/2)^n u(n-1) + 2^n \cancelto{0}{u}(-n) - \cancelto{0}{(1/2)^{n-1}} u(n-2) - 2^{n-1} u(-n+1)$$

$$h(n) = (1/2)^n - 2^{n-1}$$

$n=1$

$$h(1) = (1/2)^1 - 2^{1-1} = \frac{1}{2} - 2^0 = \frac{1}{2} - 1 = -0.5$$

$$\boxed{h(1) = -0.5}$$

(ii) $\boxed{n>1}$ 1st & 3rd

$$h(n) = (1/2)^n \overset{✓}{u}(n-1) + 2^n \cancelto{0}{u}(-n) - \left(\frac{1}{2}\right)^{\overset{✓}{n-1}} u(n-2)$$
$$- 2^{n-1} \cancelto{0}{u}(-n+1)$$

$$h(n) = \left(\frac{1}{2}\right)^n - \left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^n - \left[\left(\frac{1}{2}\right)^n \cdot \left(\frac{1}{2}\right)^{-1}\right]$$

$$\boxed{h(n) = -\left(\frac{1}{2}\right)^n} \qquad = \left(\frac{1}{2}\right)^n - \left[2\left(\frac{1}{2}\right)^n\right]$$

(iii) $\underline{\underline{n < 0}}$ $2^{n0} 7 4+4$

$h(n) = \left(\frac{1}{2}\right)^n \overset{0}{\cancel{u}}(n-1) + 2^n u(-n) - \left(\frac{1}{2}\right)^{n-1} \overset{0}{\cancel{u}}(n-2)$ ✓
$- 2^{n-1} u(-n+1)$ ✓

$h(n) = 2^n - 2^{n-1}$

$$= 2^n - 2^n \cdot 2^{-1}$$
$$= 2^n - \frac{2^n}{2}$$
$$= 2^n \left[1 - 1/2\right]$$
$$= 2^n \left[1/2\right] = \frac{2^n}{2^1}$$
$$= \frac{\cancel{2^n}}{2} = 2^{n-\cancel{2}}$$
$$= 2^n \cdot 2^{-1} = \underline{\underline{2^{n-1}}}$$

$$\boxed{h(n) = 2^{n-1}}$$

$$\boxed{h(n) = \begin{cases} 2^{n-1} & ; n < 0 \\ 0.5 & ; n = 0 \\ -0.5 & ; n = 1 \\ -(1/2)^n & ; n > 1 \end{cases}}$$

Evaluate the operations

① ~~x(t)~~ $\left[e^{-t} u(t)\right] \cdot \delta(t-2)$

$$= e^{-2} u(2) \cdot \delta(t-2)$$
$$= \underline{\underline{e^{-2} \delta(t-2)}}$$

$: x(t) \cdot \delta(t - t_0)$
$= x(t_0) \cdot \delta(t - t_0)$

② $\int_{-\infty}^{\infty} e^{-t} \delta(t-2)\, dt$ is of the form $\int_{-\infty}^{\infty} x(t) \cdot \delta(t-t_0)\, dt$
$= x(t)\big|_{t=t_0}$

$$= e^{-t}\big|_{t=2} = \underline{\underline{e^{-2}}}$$

③ $e^{-t} u(t) * \delta(t-2)$ wkt $x(t) * \delta(t) = x(t)$
$x(t) * \delta(t-t_0) = x(t-t_0)$

$$= e^{-t} \cdot u(t)\big|_{t = t-t_0} = \underline{\underline{e^{-(t-2)} \cdot u(t-2)}}$$

Module - 3

Fourier Representations of signals

CTFS Properties & basic Problem

Time property	Periodic	Non-Periodic
continuous - time	Fourier series [CTFS/FS]	Fourier Transform [FT/CTFT]
Discrete - time	Discrete -Time Fourier series [DTFS]	Discrete -Time Fourier transform [DTFT]

Continuous - Time periodic signals: The Fourier series

CTFS/FS

A periodic continuous -Time signal $x(t)$ is expressed as

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)\, e^{jk\omega_0 t} \longrightarrow ①$$

where

$$X(k) = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-jk\omega_0 t}\, dt \rightarrow ②$$

$x(t) \rightarrow$ periodic CT signal, with a fundamental period 'T' & fundamental freq $\boxed{\omega_0 = \frac{2\pi}{T}}$ rad/sec

$X(k)$ = Fourier series coefficient of $x(t)$

eq - ① is synthesis ~~eq~~ equation

eq ② is analysis eqn

$$x(t) \xleftrightarrow{FS} X(k).$$

$|X(k)|$ = Magnitude Spectrum

$\measuredangle X(k)$ = Phase Spectrum
arg $X(k)$

Properties of FS

1. Linearity :

If $x(t) \xleftrightarrow{FS} X(K)$

& $y(t) \xleftrightarrow{FS} Y(K)$

then

$$z(t) = a\,x(t) + b\,y(t) \xleftrightarrow{FS} Z(K) = a\,X(K) + b\,Y(K)$$

Proof:

$$X(K) = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-jKw_0 t}\, dt$$

$$Y(K) = \frac{1}{T}\int_{\langle T\rangle} y(t)\, e^{-jKw_0 t}\, dt$$

$$\therefore\quad Z(K) = \frac{1}{T}\int_{\langle T\rangle} z(t)\, e^{-jKw_0 t}\, dt$$

$$= \frac{1}{T}\int_{\langle T\rangle} [a\,x(t) + b\,y(t)]\, e^{-jKw_0 t}\, dt$$

$$= \boxed{\frac{1}{T}\, a \int_{\langle T\rangle} x(t)\, e^{-jKw_0 t}\, dt} + \boxed{\frac{1}{T}\, b \int_{\langle T\rangle} y(t)\, e^{-jKw_0 t}\, dt}$$

$$Z(K) = a\,X(K) + b\,Y(K)$$

② ~~St~~ Time-shift

If $x(t) \xleftrightarrow{FS} X(K)$

Then $w(t) = x(t-t_0) \xleftrightarrow{FS} W(K) = e^{-jK\omega_0 t_0} X(K)$

Proof:

$$X(K) = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-jK\omega_0 t}\, dt$$

$$\therefore W(K) = \frac{1}{T}\int_{\langle T\rangle} w(t)\, e^{-jK\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{\langle T\rangle} x(t-t_0)\, e^{-jK\omega_0 t}\, dt$$

put $t - t_0 = m$, then $\therefore t = m + t_0$

$$W(k) = \frac{1}{T}\int_{\langle T\rangle} x(m)\, \underbrace{e^{-jK\omega_0 (m+t_0)}}_{e^{-jK\omega_0 m} \cdot e^{-jK\omega_0 t_0}}\, dm$$

$= ~~e~~$

$$\therefore e^{-jK\omega_0 t_0} \underbrace{\frac{1}{T}\int_{\langle T\rangle} x(m)\, e^{-jK\omega_0 m}\, dm}$$

$$W(K) = e^{-jK\omega_0 t_0}\, X(K)$$

②

③ frequency shift

$$\text{If} \quad x(t) \xleftrightarrow{FS} X(K)$$

$$\text{then} \quad g(t) = e^{jK_0 w_0 t} x(t) \xleftrightarrow{FS} G(K) = X(K-K_0)$$

Proof:- $X(K) = \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-jKw_0 t} \, dt$

$$\therefore \quad G(K) = \frac{1}{T} \int_{\langle T \rangle} g(t) e^{-jKw_0 t} \, dt$$

$$= \frac{1}{T} \int \underbrace{e^{+jK_0 w_0 t} \quad x(t) \, e^{-jKw_0 t}}_{e^{-jw_0(K-K_0)t}} \, dt$$

$$= \frac{1}{T} \int x(t) \, e^{-jw_0(K-K_0)t} \, dt$$

$$\left[\begin{array}{l} X(K) \longleftrightarrow \frac{1}{T} \int x(t) \, e^{-jw_0 K} dt \\ \therefore \quad X(K-K_0) \longleftrightarrow \frac{1}{T} \int x(t) \, e^{-jw_0(K-K_0)} dt \end{array} \right]$$

$$\boxed{G(K) = X(K-K_0)}$$

④ scaling

If $x(t) \xleftrightarrow{FS} X(K)$

then $z(t) = x(at) \xleftrightarrow{FS} Z(K) = X(K)$; $a > 0$

Proof: If $x(t)$ is periodic then $z(t) = x(at)$ is also periodic.

If $x(t)$ has fundamental period T

Then $z(t)$ has fundamental period $\frac{T}{a}$

fundamental freq of $x(t)$ is w_0 rad/sec

then for $z(t) = x(at)$ is aw_0 (rad/sec)

$$X(K) = \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-jKw_0 t} dt$$

$$\therefore Z(K) = \frac{1}{(T/a)} \int_{\langle T/a \rangle} z(t) e^{-jKaw_0 t} dt$$

$$= \frac{a}{T} \int_{\langle T/a \rangle} x(at) e^{-jKaw_0 t} dt$$

put $at = m$ then $dt = \frac{1}{a} dm$

$$\therefore Z(K) = \frac{a}{T} \int_{\langle T/a \rangle} x(m) e^{-jKw_0 m} \frac{1}{a} dm$$

$$\therefore Z(K) = X(K)$$

⑤ Time-differentiation :

If $x(t) \xleftrightarrow{FS} X(K)$

then $\frac{dx(t)}{dt} \xleftrightarrow{FS} jKW_0 X(K)$

Proof: WKT ICTFS

$$x(t) = \sum_{K=-\infty}^{\infty} X(K) e^{jKW_0 t} \quad \text{— ①}$$

differentiating both the sides WRT time 't' we get

$$\frac{dx(t)}{dt} = \frac{d}{dt}\left[\sum_{K=-\infty}^{\infty} X(K) e^{jKW_0 t}\right]$$

Changing the order of differentiation & summation,

$$\frac{dx(t)}{dt} = \sum_{K=-\infty}^{\infty} X(K) \frac{\partial}{\partial t} e^{jKW_0 t}$$

$$= \sum_{K=-\infty}^{\infty} X(K) e^{jKW_0 t} \cdot (jKW_0)$$

$$\frac{dx(t)}{dt} = \sum_{K=-\infty}^{\infty} jKW_0 X(K) e^{jKW_0 t} \quad \text{— ②}$$

Comparing eq ② with eq ①

$$\frac{dx(t)}{dt} \xleftrightarrow{FS} jKW_0 X(K)$$

⑥ Convolution

If $x(t) \xleftrightarrow{FS} X(K)$

& $y(t) \xleftrightarrow{FS} Y(K)$

then

$z(t) = x(t) \circledast y(t) \xleftrightarrow{FS} Z(K) = TX(K) \cdot Y(K)$

$\circledast$ denotes periodic convolution

Proof:

$$X(K) = \frac{1}{T} \int_{<T>} x(t) e^{-jk\omega_0 t} dt$$

$$Y(K) = \frac{1}{T} \int_{<T>} y(t) e^{-jk\omega_0 t} dt$$

$$\therefore Z(K) = \frac{1}{T} \int_{<T>} z(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T} \int_{<T>} [x(t) \circledast y(t)] e^{-jk\omega_0 t} dt$$

using the definition of convolution.

$$= \frac{1}{T} \int_{<T>} \left[\int_{\ell=<T>} x(\ell) * y(t-\ell) d\ell \right] e^{-jk\omega_0 t} dt$$

changing the order of integration

$$Z(K) = \frac{1}{T} \int_{\ell:<T>} x(\ell) \int_{t:<T>} y(t-\ell) e^{-jk\omega_0 t} d\ell \, dt$$

Put $t - l = m$ $dt = dm$ then

$$Z(K) = \frac{1}{T} \int_{l=\langle T\rangle} x(l) \int_{m=\langle T\rangle} y(m) \underbrace{e^{-jKw_0(m+l)}}_{e^{-jKw_0 m} \cdot e^{-jKw_0 l}} \, dl \, dm$$

$$= \frac{1}{T} \int_{l=\langle T\rangle} x(l) e^{-jKw_0 l} \, dl \cdot \int_{m=\langle T\rangle} y(m) e^{-jKw_0 m} \, dm$$

$$= \frac{1}{\cancel{T}} \left[T X(K) \cdot \cancel{T} Y(K) \right]$$

$$Z(K) = T X(K) \cdot Y(K)$$

∴ convolution in time domain is transformed to multiplication of FS coefficients

⑦ **Modulation**

If $x(t) \xleftrightarrow{FS} X(K)$

& $y(t) \xleftrightarrow{FS} Y(K)$

then $z(t) = x(t) \cdot y(t) \xleftrightarrow{FS} Z(K) = X(K) * Y(K)$

Proof.

$$Z(K) = \frac{1}{T} \int_{\langle T\rangle} z(t) e^{-jKw_0 t} \, dt$$

$$= \frac{1}{T} \int_{\langle T\rangle} x(t) \cdot y(t) e^{-jKw_0 t} \, dt \quad \text{——} ①$$

WKT

$$x(t) = \sum_{l=-\infty}^{\infty} X(l) e^{jlw_0t} \quad \text{——} ②$$

defn of IFS

Substituting eq ② in eq ① we get

$$Z(K) = \frac{1}{T}\int_{\langle T\rangle}\left[\sum_{l=-\infty}^{\infty} X(l) e^{jw_0lt}\right] y(t) e^{-jkw_0t} dt$$

changing the order of summation & integration we get

$$Z(K) = \frac{1}{T}\left[\sum_{l=-\infty}^{\infty} X(l) \int_{\langle T\rangle} y(t) e^{-jw_0lt} \underbrace{e^{-jKw_0t}}_{e^{-j(K-l)w_0t}} dt\right]$$

$$= \frac{1}{T}\left[\sum_{l=-\infty}^{\infty} X(l) \int_{\langle T\rangle} y(t) e^{-j(K-l)w_0t} dt\right]$$

using freq shift property

$$X(k) = X(K-K_0) = \frac{1}{T}\int x(t) e^{-j\omega_0(K-K_0)t} dt$$

$$= \sum_{l=-\infty}^{\infty} X(l) \cdot Y(K-l)$$

$$Z(K) = X(K) * Y(K)$$

⑧ Parseval's theorem

If $x(t) \xleftrightarrow{FS} X(K)$

then $$\frac{1}{T}\int_{\langle T\rangle} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X(k)|^2 \quad — ①$$

Proof/- The LHS of above eqn ① is the average power of a periodic continous-time signal $x(t)$ with fundamental Period T.

i.e. $$P = \frac{1}{T}\int_{\langle T\rangle} |x(t)|^2 dt$$

This eqn can be written as

$$= \frac{1}{T}\int_{\langle T\rangle} x(t) \cdot x^*(t)\, dt$$

$$= \frac{1}{T}\int_{\langle T\rangle} x(t)\left[\sum_{K=-\infty}^{\infty} X^*(K)\, e^{-jKw_0t}\right] dt$$

changing the order of summation & integration we get

$$P = \frac{1}{T}\sum_{K=-\infty}^{\infty} X^*(K) \int_{\langle T\rangle} x(t)\, e^{-jKw_0t}\, dt$$

$$= \sum_{K=-\infty}^{\infty} X^*(K)\, X(K) = \sum_{K=-\infty}^{\infty} |X(K)|^2$$

$$\therefore \frac{1}{T}\int_{\langle T\rangle} |x(t)|^2 dt = \sum_{K=-\infty}^{\infty} |X(K)|^2 \;// \quad ——— ②$$

In eq ② the seqn $|X(K)|^2$ for $K=0,1,2\ldots$ is the distribution of power as a function of frequency & it is called Powerdensity spectrum of the signal $x(t)$

⑨ symmetry:-

If $x(t) \xleftrightarrow{FS} X(K)$

Then (i) If $x(t)$ is real, then $X^*(K) = X(-K)$

(ii) If $x(t)$ is real & even then

$$\text{Img}\{X(K)\} = 0$$

(iii) If $x(t)$ is real & odd, then

$$\text{Re}\{X(K)\} = 0$$

some Basic problems of CTFS

① For the signal $x(t) = \sin(\omega_0 t)$, find the Fourier series & draw its spectrum

Soln: $x(t) = \sin\omega_0 t$

WKT $\sin x = \frac{e^{x} - e^{-x}}{2j}$

$$= \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$$

$$x(t) = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}$$

$$= \frac{1}{2j}e^{j(1)\omega_0 t} - \frac{1}{2j}e^{j(-1)\omega_0 t} \quad \text{——} ①$$

[express it terms of $X(K)e^{j\underline{K}\omega_0 t}$]

⑥

$$x(t) = \underset{X(1)}{\frac{1}{2j}}\, e^{j\underset{K=1}{(1)}\omega_0 t} - \underset{X(-1)}{\frac{1}{2j}}\, e^{j\underset{K=-1}{(-1)}\omega_0 t} \longrightarrow ①$$

comparing eq ① with IFS synthesis eq ②

$$x(t) = \sum_{K=-\infty}^{\infty} X(K)\, e^{jK\omega_0 t} \longrightarrow ②$$

when we expand eq ② we get

$$X(-\infty)\, e^{j(-\infty)\omega_0 t} + \cdots + X(-2)\, e^{j(-2)\omega_0 t}$$

$$+ X(-1)\, e^{j(-1)\omega_0 t} + \underset{\overset{\|}{X(0)}}{X(0)\, e^{j(0)\omega_0 t}}$$

$$+ X(1)\, e^{j(1)\omega_0 t} + X(2)\, e^{j(2)\omega_0 t} + \cdots$$

we get

$X(1) = \frac{1}{2j}$ $\quad X(-1) = -\frac{1}{2j}$

~~$X(0) = 0$~~

and $X(K) = 0$ for $K \neq \pm 1$

$X(K)$ components are

$X(1) = \frac{1}{2j}$ & $X(-1) = -\frac{1}{2j}$

<u>to find & plot magnitude & phase</u>

$X(1) = \frac{1}{2j}$ $X(-1) = -\frac{1}{2j}$

X'ly ÷ by j [we want to express in term of a+jb]

$X(1) = \frac{j}{2j \times j}$

$= \frac{j}{2 \cdot j^2}$

$X(1) = -\frac{1}{2} \cdot j$

$X(1) = -0.5j$

$X(-1) = \frac{-j}{2j \times j}$

$= \frac{-j}{2j^2}$ $\therefore j^2 = -1$

$X(-1) = \frac{1}{2}j$

$X(-1) = +0.5j$

~~$X(-1) = 0$~~

$X(1) = 0 - 0.5j$ & $X(-1) = 0 + 0.5j$

Magnitude :- $|X(k)| = \sqrt{a^2+b2}$

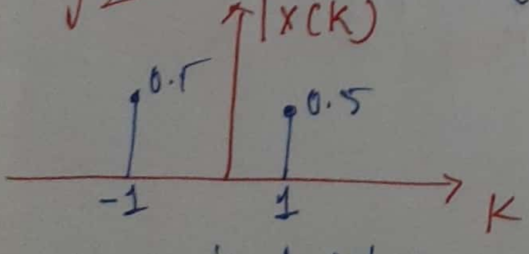
$\boxed{|X(1)|} = \sqrt{0^2+(-0.5)^2} = \boxed{0.5}$

$\boxed{|X(-1)|} = \sqrt{0^2+(0.5)^2} = \boxed{0.5}$

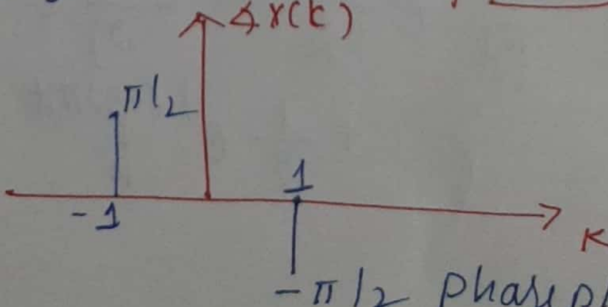
Phase $\measuredangle X(k) = \tan^{-1}(b/a)$

$\boxed{\measuredangle X(1)} = \tan^{-1}\left(\frac{-0.5}{0}\right) = \tan^{-1}(-\infty) = \boxed{-\pi/2}$

$\boxed{\measuredangle X(-1) =} \tan^{-1}\left(\frac{0.5}{0}\right) = \tan^{-1}(\infty) = \boxed{\pi/2}$



Magnitude plot



phase plot

② Evaluate the FS representation for the signal

$x(t) = \sin(2\pi t) + \cos(3\pi t)$

Sol^n/ $x(t) = \sin(\overset{w_{01}t}{2\pi t}) + \cos(\overset{w_{02}t}{3\pi t})$ — ①

$= x_1(t) + x_1(t)$

$x_1(t)$ - angulal freq $w_{01} = 2\pi$

$x_2(t)$ angularfreq $w_{02} = 3\pi$

∴ the angular freq of $x(t)$

$= g.c.d(w_{01}, w_{02})$

$= g.c.d(2\pi, 3\pi)$

$= \underline{\underline{\pi}}$

Again we need to compare with synthesis eqn i.e

~~$x(t) = \sum_{K=-\infty}^{\infty}$~~

$$x(t) = \sum_{K=-\infty}^{\infty} X(K)\, e^{jKw_0t}$$

so express eq① in to exponential form

$$x(t) = \frac{1}{2j} e^{j\underline{2}\pi t} - \frac{1}{2j} e^{-j\underline{2}\pi t} + \frac{1}{2} e^{j\underline{3}\pi t} + \frac{1}{2} e^{-j\underline{3}\pi t}$$

(e^{jKw_0t})

$$= \frac{1}{2j} e^{j\underset{\underset{K}{\downarrow}}{(2)}\pi t} - \frac{1}{2j} e^{j\underset{\underset{K}{\downarrow}}{(-2)}\pi t} + \frac{1}{2} e^{j\underset{\underset{K}{\downarrow}}{(3)}\pi t}$$

$$+ \frac{1}{2} e^{j\underset{\underset{K}{\downarrow}}{(-3)}\pi t}$$

$X(2) = \frac{1}{2j}$; $X(-2) = -\frac{1}{2j}$

$X(3) = \frac{1}{2}$; $X(-3) = \frac{1}{2}$

& $X(K) = 0$ for $K \neq \pm 2, \pm 3$

$|X(2)| = 0 - 0.5j$ $|X(-2)| = 0 + 0.5j$ $|X(3)| = 0.5$

$|X(-3)| = 0.5$

Magnitude values

$|X(2)| : 0.5$ $|X(-2)| = 0.5$ $|X(3)| = 0.5$

$|X(-3)| = 0.5$

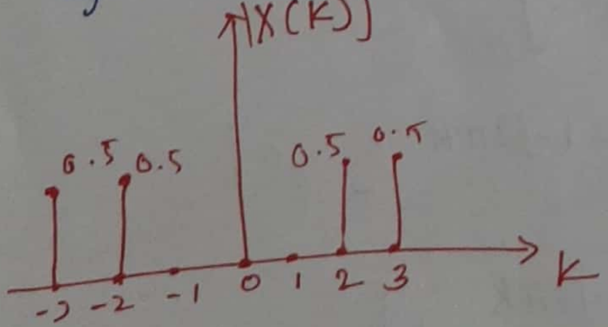
phase values

$\measuredangle X(2) = \tan^{-1}\left(\frac{-0.5}{0}\right) = \tan^{-1}(-\infty) = -\pi/2$

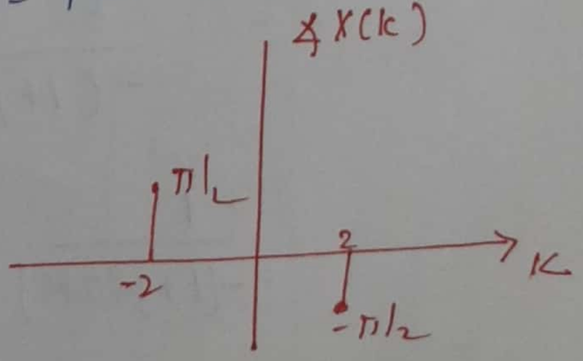
$\measuredangle X(-2) = \tan^{-1}\left(\frac{0.5}{0}\right) = \tan^{-1}(\infty) = \pi/2$

$\measuredangle X(3) = \tan^{-1}\left(\frac{0}{0.5}\right) = \tan^{-1}(0) = 0$

$\measuredangle X(-3) = \tan^{-1}\left(\frac{0}{0.5}\right) = \tan^{-1}(0) = 0$

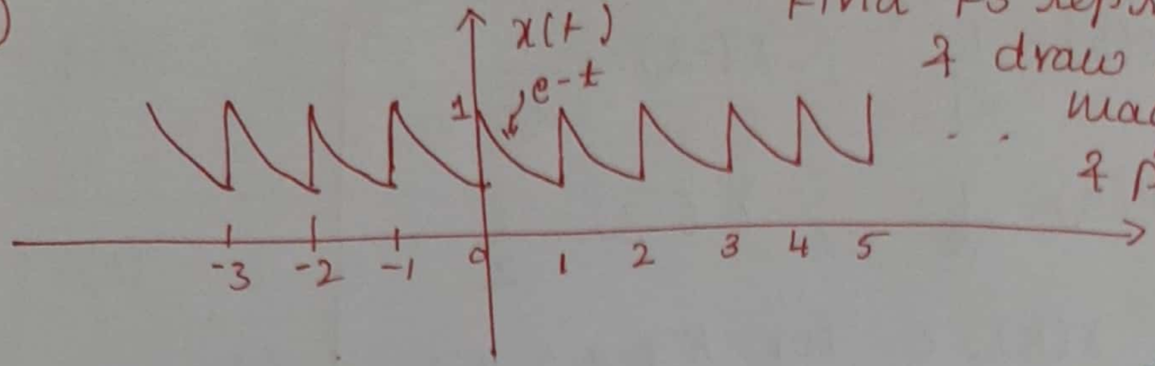


Magnitude plot



phase plot

③



Find FS representation & draw its magnitude & phase spectra

Soln! using Analysis eqn $x(t) = e^{-t}$

$$X(K) = \frac{1}{T}\int_{<T>} x(t)\, e^{-jKw_0t}\, dt$$

here $\boxed{T = 1}$ & $\boxed{w_0 = \frac{2\pi}{T} = 2\pi}$

$$\therefore X(K) = \frac{1}{1}\int_{t=0}^{1} e^{-t} \cdot e^{-jK2\pi t}\, dt$$

$$= \int_{t=0}^{1} e^{-(1+j2\pi K)t}\, dt$$

$$= \left.\frac{e^{-(1+j2\pi K)t}}{-(1+j2\pi K)}\right]_0^1$$

$$= \frac{1}{-(1+j2\pi K)}\left[e^{-1-j2\pi K} - e^{0}\right]$$

$$= \frac{-1}{(1+j2\pi K)}\left[e^{-1-j2\pi K} - 1\right]$$

$$= \frac{1 - e^{-1-j2\pi K}}{1+j2\pi K} = \frac{1 - e^{-1}.e^{-j2\pi K}}{1+j2\pi K}$$

$$X(K) = \frac{1 - e^{-1} e^{-j2\pi K}}{1 + j2\pi K}$$

$$\boxed{X(K) = \frac{1 - e^{-1}}{1 + j2\pi K}}$$

$e^{-j2\pi K} = \cos K2\pi - j\,\overset{0}{\cancel{\sin K2\pi}}$

K = integer.

$\sin K2\pi = 0$ for ∀ values of K

$\cos K2\pi = 1$ for K = 1 & its multiples

$\therefore \boxed{e^{-j2\pi K} = 1 \;\forall K}$

to find & plot magnitude plot

$$\boxed{|X(K)| = \frac{|1 - e^{-1}|}{\sqrt{1^2 + (2\pi K)^2}}}$$

Phase value

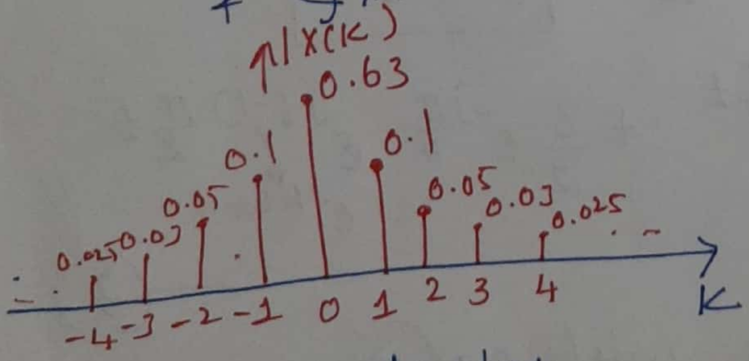
$$\measuredangle X(K) = \frac{0}{\tan^{-1}\left(\frac{2\pi K}{1}\right)}$$

← Real value phase = 0

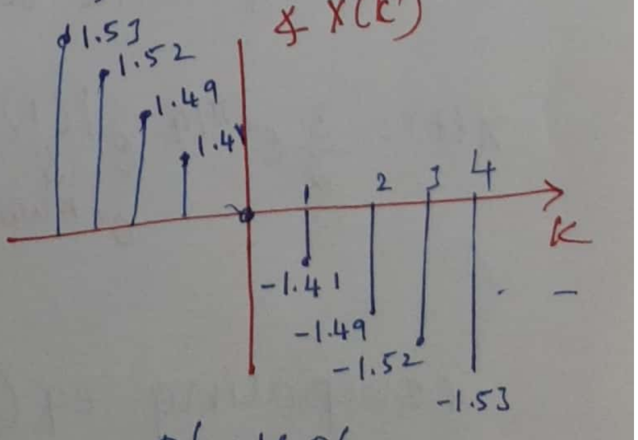
Keep Calculators in radians

$$\boxed{\measuredangle X(K) = -\tan^{-1}(2\pi K)}$$

for different values of K get |X(K)| & ∡X(K) values   K = 0, ±1, ±2, . . .



Magnitude plot



phase plot

④ $x(t) = 3\cos\left(\frac{\pi}{2}t + \frac{\pi}{4}\right)$ —— ①

soln: It is of the form $A\cos(\underbrace{w_0 t + \phi}_{=\theta})$

$w_0 = \frac{\pi}{2} \quad \phi = \frac{\pi}{4}$

compare with synthesis eqn

$$x(t) = \sum_{K=-\infty}^{\infty} X(K)\, e^{jKw_0 t} \quad \text{—— ②}$$

so express eq ① in terms of exponential form $\quad x = \frac{\pi}{2}t + \frac{\pi}{4} \quad \cos x = \frac{e^{jx} + e^{-jx}}{2}$

$$x(t) = 3\left[\frac{e^{j(\frac{\pi}{2}t + \frac{\pi}{4})} + e^{-j(\frac{\pi}{2}t + \frac{\pi}{4})}}{2}\right]$$

$$= \frac{3}{2}\underbrace{e^{j(\frac{\pi}{2}t + \frac{\pi}{4})}} + \frac{3}{2}\underbrace{e^{-j(\frac{\pi}{2}t + \frac{\pi}{4})}}$$

$$= \frac{3}{2}e^{j\pi/4}\, e^{j\pi/2\, t} + \frac{3}{2}e^{-j\pi/4} \cdot e^{-j\pi/2\, t}$$

we need to write in term of e^{jKw_0t}

& since $w_0 = \frac{\pi}{2}$ so $\boxed{e^{jK\frac{\pi}{2}t}}$

$$x(t) = \frac{3}{2}e^{j\pi/4}\, \underset{e^{jKw_0t}}{e^{j(1)\frac{\pi}{2}t}} + \frac{3}{2}e^{-j\frac{\pi}{4}}\, \underset{e^{jKw_0t}}{e^{j(-1)\frac{\pi}{2}t}} \quad \text{—— ③}$$

comparing eq ③ with eq ①

$X(1) = \frac{3}{2} e^{j\frac{\pi}{4}}$ $X(-1) = \frac{3}{2} e^{-j\pi/4}$

$X(K) = 0$ for $K \neq \pm 1$

Magnitude value

$\underset{\substack{\uparrow \\ \text{magnitude}}}{A} e^{j\overset{\text{phase}}{x}}$

$|X(1)| = \frac{3}{2}$

$|X(-1)| = \frac{3}{2}$

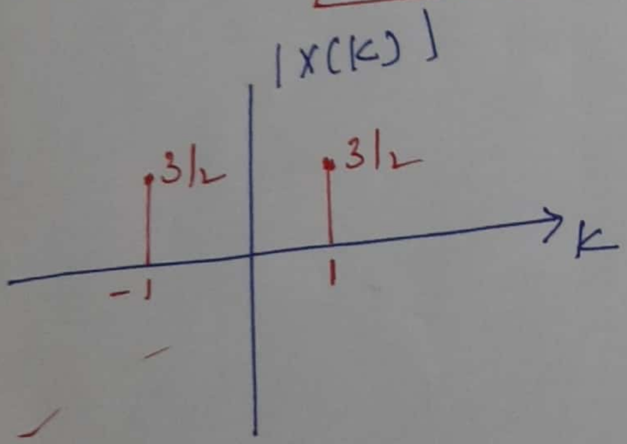
phase value

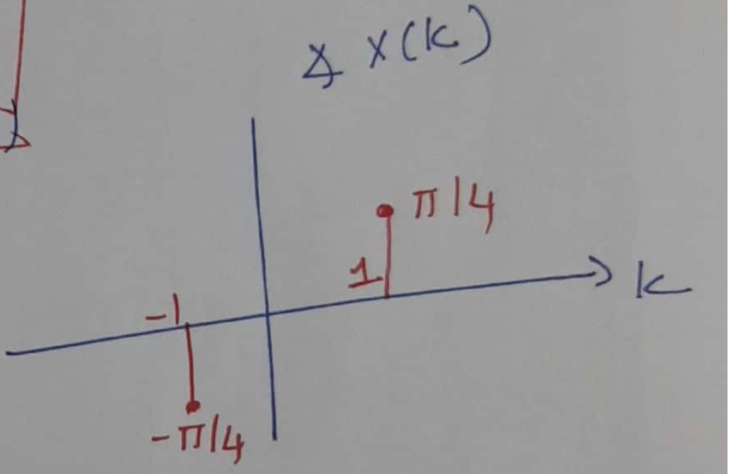
$\measuredangle X(+1) = \pi/4$

$\measuredangle X(-1) = -\pi/4$

$\underset{\substack{\uparrow \\ \text{magnitude}}}{A} e^{j\overset{\leftarrow \text{phase}}{x}}$

general Form



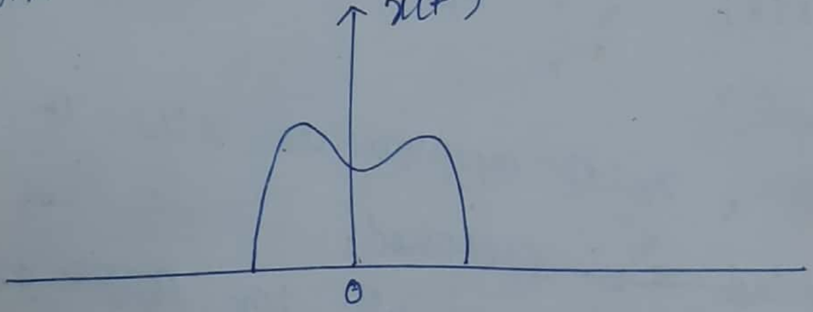


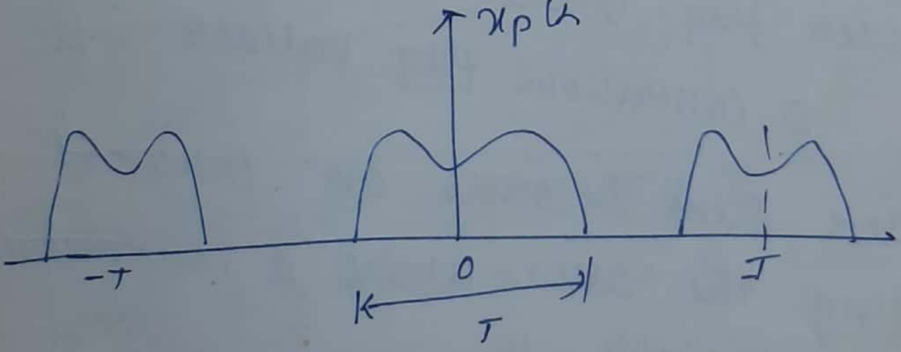
(UNIT- 5) Module- 4 → Xerox 02 ①

FOURIER REPRESENTATION FOR SIGNALS - 2

FOURIER Transforms :-

- The FT is used to represent a continuous-time non-periodic signal as a superposition of complex sinusoids.
- consider a non-periodic (signal) fun $x(t)$ which can be treated as a periodic signal $x_p(t)$ with a period T as shown below





$$\therefore \quad \text{If} \quad \lim_{T\to\infty} x_p(t) = x(t)$$

- By FS any periodic signal $x_p(t)$ can be represented by

exponential FS → $$x_p(t) = \sum_{K=-\infty}^{\infty} X(K)\, e^{jKw_0 t} \longrightarrow ①$$

&

coefficient X(k) → $$X(K) = \frac{1}{T}\int_T x_p(t)\, e^{-jKw_0 t}\,dt \longrightarrow ②$$

* Let us ~~defining~~

* Let us define (i) $\omega_0 = \frac{2\pi}{T}$ (ii) $\omega_K = \frac{2\pi K}{T} = \omega_0 \cdot K$

(iii) $X(j\omega) = X(\omega) = T\,X(K)$ ∴ $X(K) = \frac{X(\omega)}{T}$

* substituting these relation in eq ① & ②
(using)

$$x_p(t) = \sum_{K=-\infty}^{\infty} \frac{X(\omega)}{T} e^{j\omega_K t}$$

&

$$X(\omega) = T\cdot X(K) = \int_T x_p(t)\, e^{-j\omega_K t}\, dt$$

* as $T \to \infty$, $x_p(t)$ approaches $x(t)$. &,
~~If the period T is stretched~~
& discrete freq variable ω_K approaches a continuous freq variable ω

& discrete sum becomes an integral defining the area under a continuous freq variable ω

* ∴ $$\boxed{x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega} \to ③$$

$$\boxed{T = \frac{2\pi}{\omega_0}}$$

$$\left\{ x_p(t) = \sum_{K=-\infty}^{\infty} \frac{X(\omega)}{2\pi/\omega_0} e^{-j\omega_K t} \right.$$

$T \to \infty$, $x_p(t) \to x(t)$

$\omega_K \to \omega$

$\omega_0 \to d\omega$

& $\Sigma = \int$]

Integra

If

Then

$y(t) =$

proof

& $\lim_{T\to\infty} T\,X(K) = X(\omega)$

$$\boxed{\therefore X(j\omega) = X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt} \rightarrow ④$$

$[T \to \infty, \quad x_p(t) = x(t)$
$\omega_K \to \omega]$

$X(j\omega)$ or $X(\omega)$
$\Downarrow$
FT of $x(t)$

$$\boxed{x(t) \xleftrightarrow{FT} X(\omega)} \rightarrow ⑤$$

Dirichlet conditions for Fourier Transform

Any aperiodic signal $x(t)$ has its FT if it satisfies the following conditions.

(i) $x(t)$ must be absolutely integrable

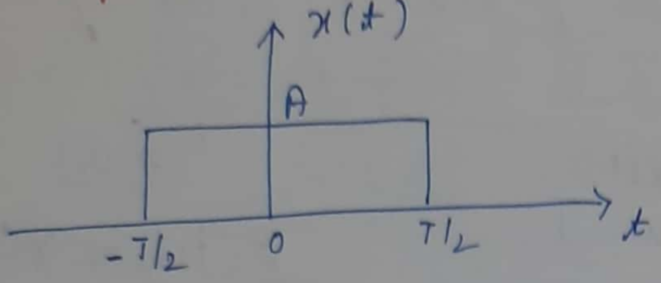
i.e. $$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

(ii) $x(t)$ ~~has~~ must have a finite no of maxima & minima in any finite interval

(iii) $x(t)$ must have finite no of discontinuities in any finite interval

(iv) The size of each discontinuity is finite.

(1) Find the Fourier Transforms of the signal & plot its magnitude & phase plot



$$x(t) = \begin{cases} A, & -T/2 \le t \le T/2 \\ 0, & e.w. \end{cases}$$

$$F\{x(t)\} = X(j\omega) = X(\omega) \triangleq \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$X(\omega) = \int_{-T/2}^{T/2} A e^{-j\omega t} dt$$

$$= \left[\frac{A e^{-j\omega t}}{-j\omega} \right]_{-T/2}^{T/2}$$

$$= \frac{A}{-j\omega} \left[e^{-j\omega\frac{T}{2}} - e^{j\omega\frac{T}{2}} \right]$$

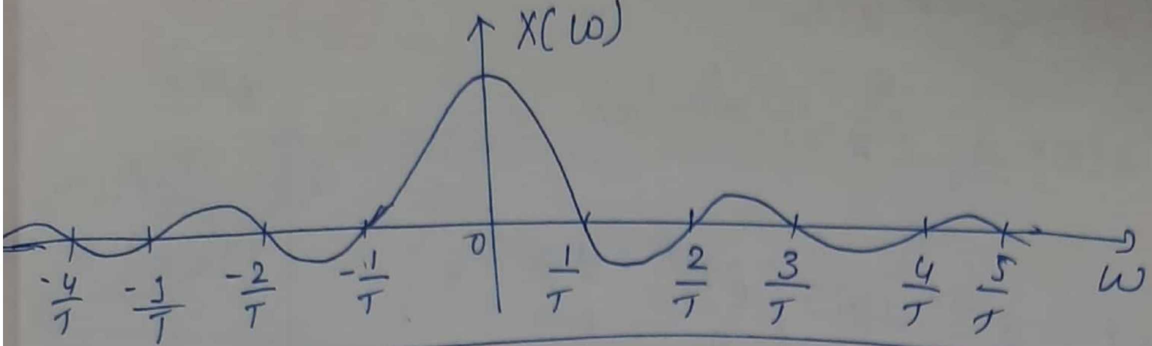
$$= \frac{2A}{\omega} \left[\frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2j} \right]$$ × & ÷ by 2

$$\boxed{\text{sinc}\, x = \frac{\sin\pi}{\pi}}$$

$$X(j\omega) = 2A \frac{\sin(\omega T/2)}{\omega}$$ WKT $\omega = 2\pi f$

$$= 2A \frac{\sin(2\pi f T/2)}{2\pi f} = \frac{AT \sin(\pi f T)}{\pi f T}$$ $\times^{ly}$ & ÷ by T

$$\boxed{X(\omega) = AT\, \text{sinc}(fT)}$$



$$\boxed{A\,\mathrm{rec}\left(\frac{t}{T}\right) \xleftrightarrow{FT} AT\,\mathrm{sinc}(fT)}$$

magnitude spect

$$|X(j\omega)| = 2A\left|\frac{\sin(\omega T/2)}{\omega}\right|$$

$$=$$

$$\angle X(j\omega) = 0 \quad \text{if} \quad \mathrm{sinc}\left(\frac{\omega T}{2}\right) > 0$$

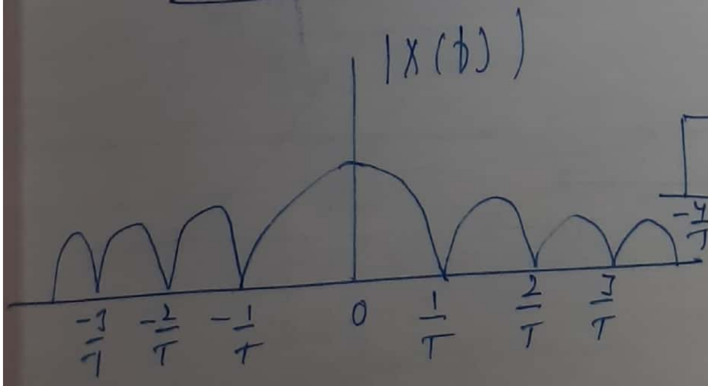
$$\pm\pi \quad \mathrm{sinc}\left(\frac{\omega T}{2}\right) < 0.$$

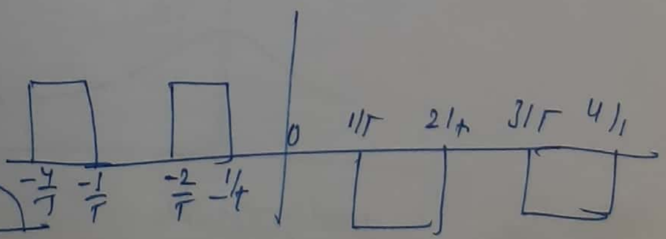
phase spectrum is

$$\arg[X(j\omega)] = \begin{cases} 0, & \frac{\sin(\omega T/2)}{\omega} > 0 \\ \pi & \frac{\sin(\omega T/2)}{\omega} < 0 \end{cases}$$

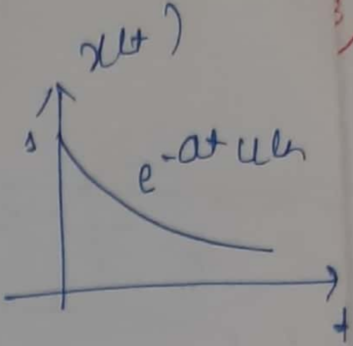
$$\mathrm{sinc}(fT) = \mathrm{sinc}\left(\frac{2\pi fT}{2\pi}\right) = \mathrm{sinc}\left(\frac{\omega T}{2\pi}\right)$$

$$\boxed{A\,\mathrm{rec}\left(\frac{t}{T}\right) \xleftrightarrow{FT} AT\,\mathrm{sinc}\left(\frac{\omega T}{2\pi}\right)}$$





② $x(t) = e^{-at}\, u(t)$



$$FT\{x(t)\} \triangleq \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$$

$$x(t) = e^{-at}u(t) = \begin{cases} e^{-at} & t \geq 0 \\ 0, & e.w \end{cases}$$

$$F\{x(t)\} = \int_{0}^{\infty} e^{-at}\, e^{-j\omega t} dt$$

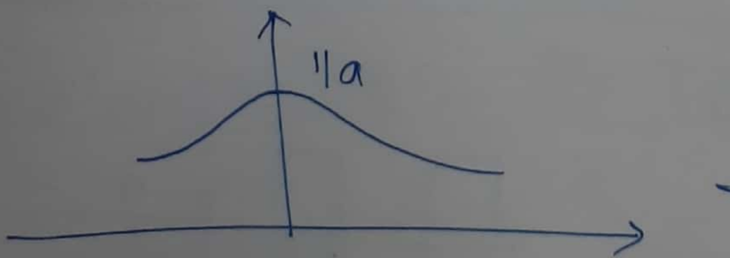
$$= \int_{0}^{\infty} e^{-(a+j\omega)t} dt$$

$$= \frac{1}{-(a+j\omega)}\, e^{-(a+j\omega)t}\Big]_{0}^{\infty}$$

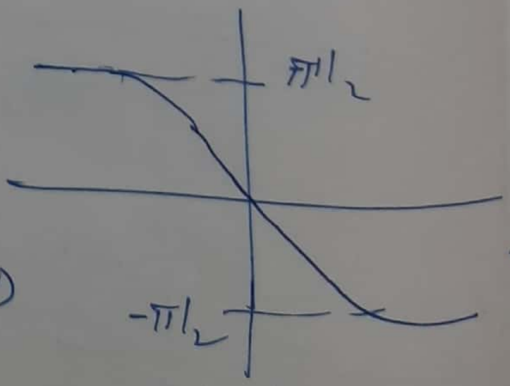
$$X(\omega) = \frac{1}{a+j\omega} = \underbrace{\frac{1}{\sqrt{a^2+\omega^2}}}_{magnitude} \underbrace{\left[-\tan^{-1}\left(\frac{\omega}{a}\right)\right]}_{phase}$$

$X(j\omega)$

$$\therefore \boxed{e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{a+j\omega}}$$



magnitude



phase

③ $e^{at} u(-t)$

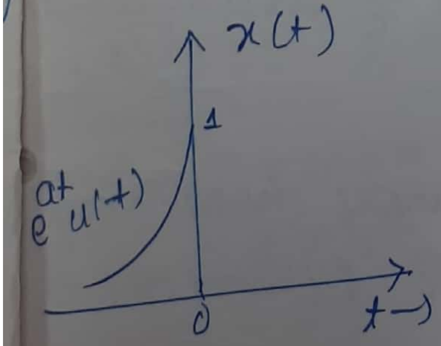
$x(t) = e^{at} u(-t)$ $\qquad F\{x(t)\} \triangleq \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

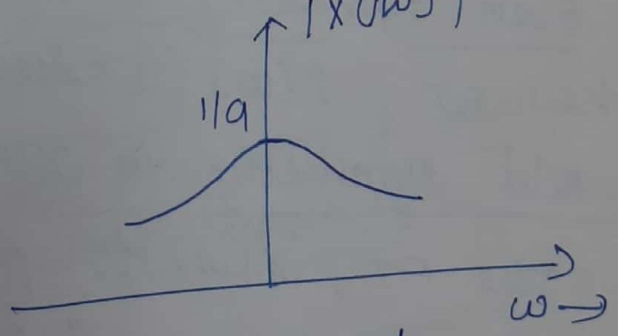
$$F\{x(t)\} = \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t} dt = \frac{1}{a-j\omega} \left. e^{(a-j\omega)} \right]_{-\infty}^{0}$$

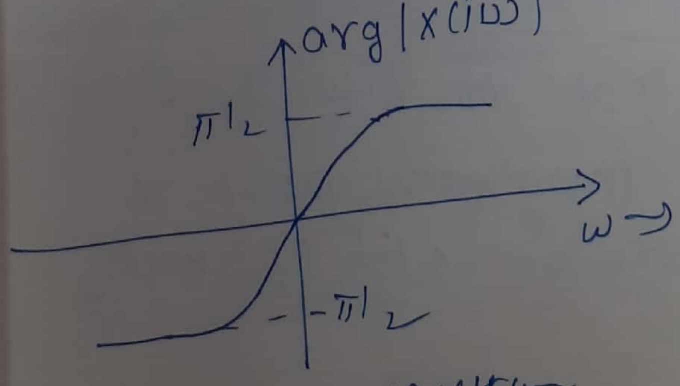
$$X(j\omega) = \frac{1}{a-j\omega}$$

$$= \underbrace{\frac{1}{\sqrt{a^2+\omega^2}}}_{mag} \underbrace{\left[-\tan^{-1}\left(\frac{-\omega}{a}\right)\right]}_{phase}$$





magnitude spectrum



phase spectrum

Magnitude & phase spectra

* The FT $X(j\omega)$ or $X(\omega)$, in general is a complex quantity & may be expressed in an exponential form as follows:-

$$X(\omega) = |X(\omega)| e^{j\phi(\omega)}.$$

* For real signals, $X(-\omega) = X^*(\omega)$.

This means that

magnitude spectrum → plot of $|X(\omega)|$ displays even symm

Phase spectrum → plot of $\phi(\omega)$ displays odd symmetry.

Important facts about spectra

1. Non periodic signals have continuous spect
2. Effect of symmetry on FT of real x(t

(a) Even symmetry in x(t) /.

the FT, $X(\omega)$ is real & even symmet

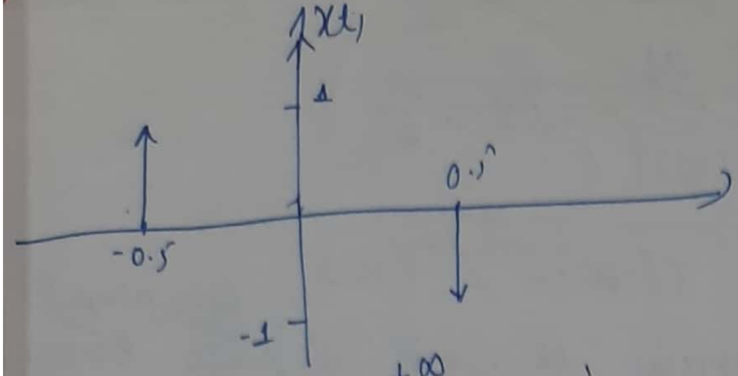
(b) odd symmetry in x(t):-

The FT, $X(\omega)$ is purely imaginary & odd symmetric

(c) No symmetry in x(t)).

The real part of $X(\omega)$ is even symmetric & imaginary part is odd symmetry

$x(t) = \delta(t+0.5) - \delta(t-0.5)$



• $x(t) \rightarrow$ is real,

$\therefore X(-\omega) = X^*(\omega)$

* magnitude spectrum displays even symmetry & phase spectrum exhibits odd ~~spe~~. sym

$$X(\omega) \triangleq \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} [\delta(t+0.5) - \delta(t-0.5)]\, e^{-j\omega t}\,dt$$

$$= e^{-j\omega t}\Big|_{t=-0.5} - e^{-j\omega t}\Big|_{t=0.5}$$

$$= e^{j0.5\omega} - e^{-j0.5\omega}$$

$$\boxed{X(\omega) = 2j\sin(0.5\omega)}$$

x(t) exhibits odd symmetry & hence X(ω) is purely imaginary.

$\tan 90 = \infty$

$\tan 0 = 0$

$$|X(\omega)| = |2\sin 0.5\omega|$$

$$\measuredangle X(\omega) = \measuredangle 2 + \measuredangle j + \measuredangle \sin(0.5\omega)$$

$$= 0 + 90^\circ +$$

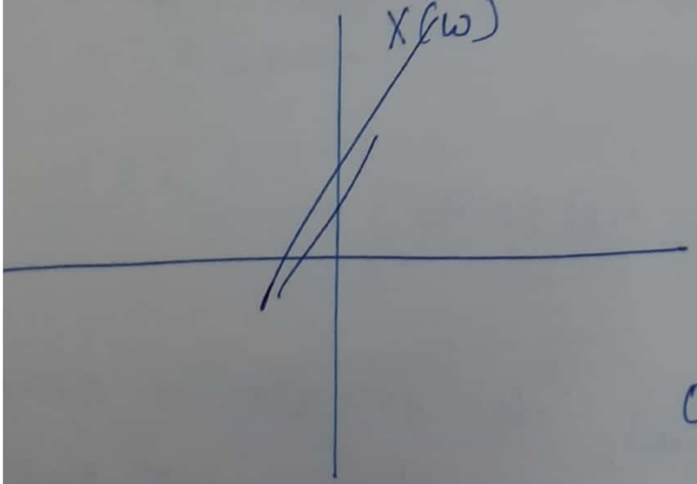
for +ve values of $\sin(0.5\omega)$, the phase is 0 radi-

−ve , , , , , , , , −π rad

∠

→ contd ③

$$\arg[X(\omega)] : \begin{cases} \frac{\pi}{2} + 0 , & \sin(0.5\omega) > 0 \\ \frac{\pi}{2} - \pi \\ = -\frac{\pi}{2} , & \sin(0.5\omega) < 0. \end{cases}$$



Zero crossing of X(ω) is obtained a

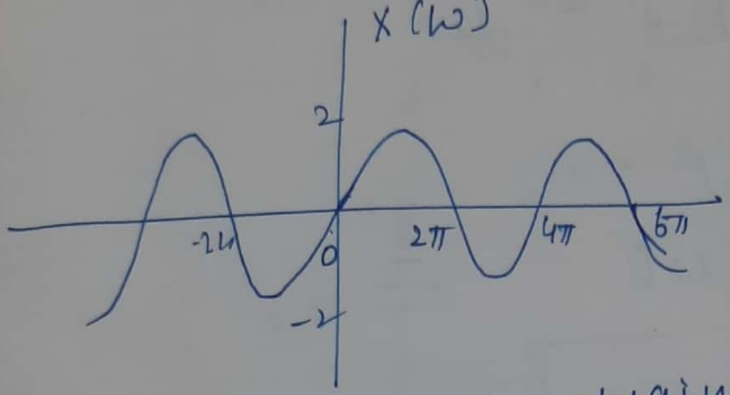
$\sin(0.5\omega) = 0$

only when

$\omega = \pm 2\pi m$

where $m \rightarrow$ an integer

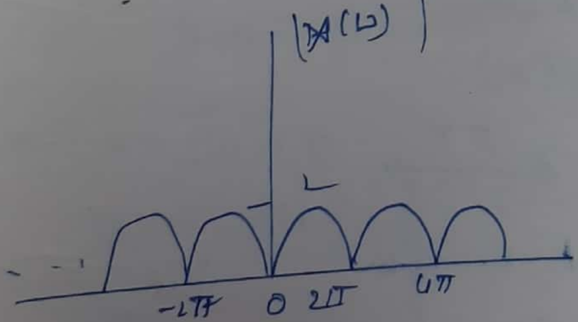
$$\angle X(\omega) = \begin{cases} \frac{\pi}{2} + 0 , & \text{for } \sin(0.5\omega) > 0 \\ \frac{\pi}{2} - \pi = -\frac{\pi}{2}, & \text{for } \sin(0.5\omega) < 0 . \end{cases}$$

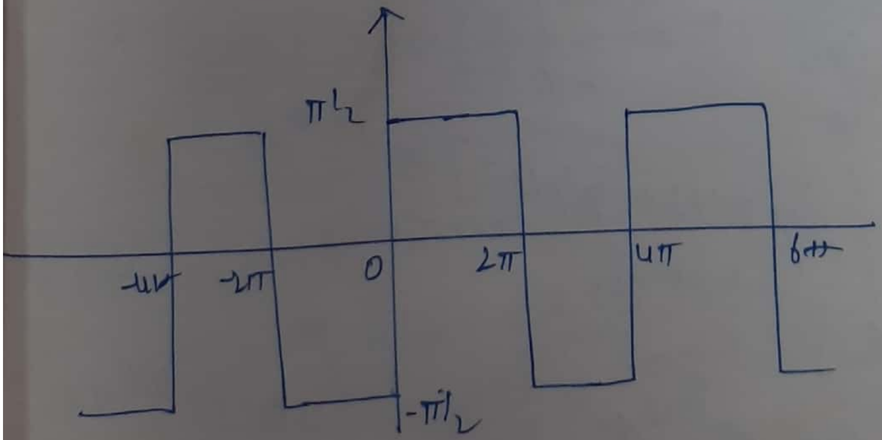


* zero-crossing is obtained

$\sin(0.5\omega) = 0$

$\omega = \pm 2\pi m$





④ unit impulse $x(t) = \delta(t)$

$$X(j\omega) \triangleq \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$$= e^{-j\omega t}\Big|_{t=0} = \underline{\underline{1}}$$

$$\boxed{\delta(t) \xleftrightarrow{FT} 1}$$

Properties of fourier Transform :

(1) **Linearity :-**

If $\quad x(t) \xleftrightarrow{FT} X(\omega)$

$\quad\quad y(t) \xleftrightarrow{FT} Y(\omega)$

then $z(t) = a x(t) + b y(t) \xleftrightarrow{FT} a X(\omega) + b Y(\omega)$

Prof

$$F\{a x(t) + b y(t)\} \triangleq \int_{-\infty}^{+\infty} z(t)\, e^{-j\omega t}\, dt$$

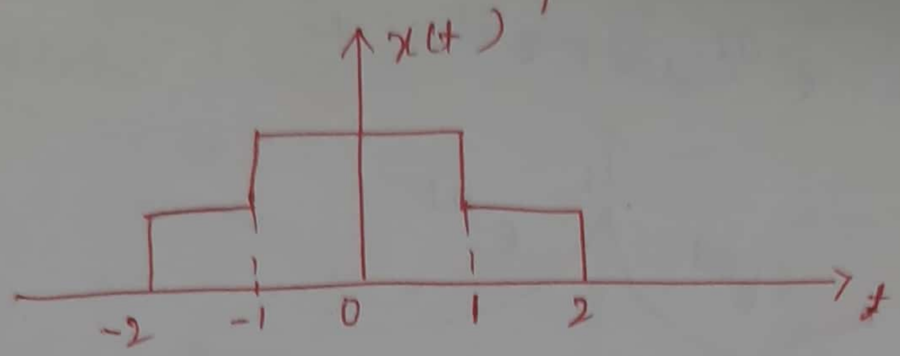
$$= Z(\omega) =$$

$$= \int_{-\infty}^{\infty} [a x(t) + b y(t)]\, e^{-j\omega t}\, dt$$

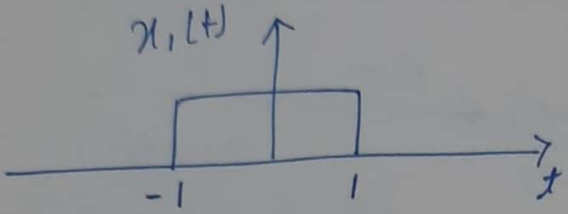
$$= a \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt + b \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= a X(\omega) + b Y(\omega)$$

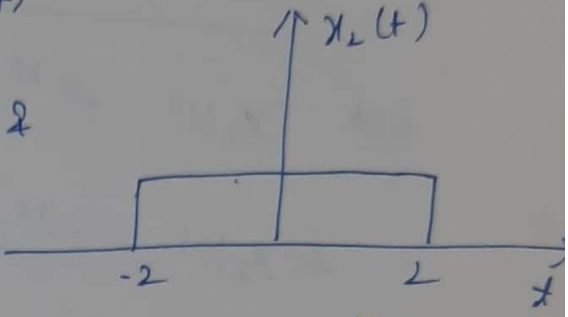
① Find the FT of the signal



solⁿ. Let $x(t) = x_1(t) + x_2(t)$



&



$x_1(t) = rect\left(\frac{t}{2}\right) \longleftrightarrow$ & $x_2(t) = rect\left(\frac{t}{4}\right)$

WKT $A\,rect\left(\frac{t}{T}\right) \xleftrightarrow{FT} AT\,sinc(fT)$

or $AT\,sinc\left(\frac{\omega T}{2\pi}\right)$

∴ $x_1(t) = rect\left(\frac{t}{2}\right) \xleftrightarrow{FT} 2\,sinc(f2)$

or $2\,sinc\left(\frac{\omega}{\pi}\right)$

& $x_2(t) = rect\left(\frac{t}{4}\right) \xleftrightarrow{FT} 4\,sinc(f4)$

or $4\,sinc\left(\frac{2\omega}{\pi}\right)$

the given signal is

$x(t): x_1(t) + x_2(t)$

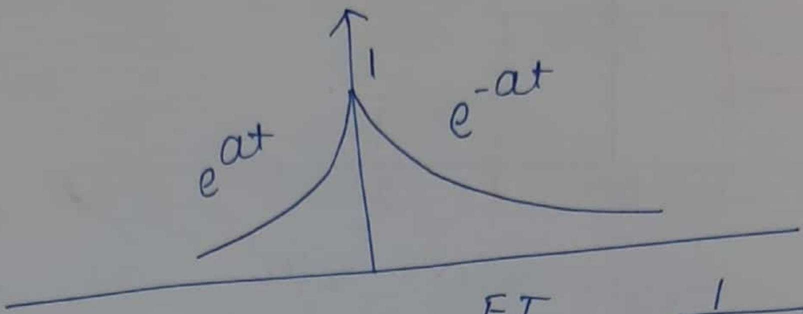
Taking FT using linearity property

$X(\omega) = X_1(\omega) + X_2(\omega)$

$$\boxed{X(\omega) = 2\,sinc\left(\frac{\omega}{\pi}\right) + 4\,sinc\left(\frac{2\omega}{\pi}\right)}$$ //

② $x(t) = e^{-a|t|}$

$$x(t) = e^{-at}u(t) + e^{at}u(-t)$$



Let $x_1(t) = e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{a+j\omega}$

& $x_2(t) = e^{at}u(-t) \xleftrightarrow{FT} \frac{1}{a-j\omega}$

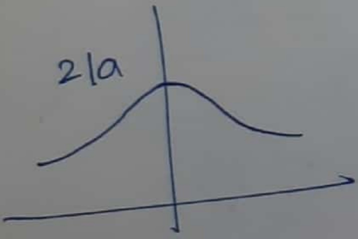
given signal $x(t) = x_1(t) + x_2(t)$

Taking FT

$$X(\omega) = X_1(\omega) + X_2(\omega)$$

$$= \frac{1}{a+j\omega} + \frac{1}{a-j\omega}$$

$$= \frac{2a}{a^2+\omega^2}$$



phase = 0.

② Time-shifting property :-

If $x(t) \xleftrightarrow{FT} X(\omega)$

then $y(t) = x(t-t_0) \xleftrightarrow{FT} Y(\omega) = e^{-j\omega t_0} X(\omega$

Proof:- $Y(\omega) \triangleq \int_{-\infty}^{+\infty} y(t) e^{-j\omega t} dt$

$$= \int_{-\infty}^{\infty} x(t-t_0) e^{-j\omega t} dt$$

Let $t - t_0 = \lambda$ then $dt = d\lambda$

$$Y(w) = \int_{-\infty}^{\infty} x(\lambda)\, e^{-jw(t_0+\lambda)}\, d\lambda$$

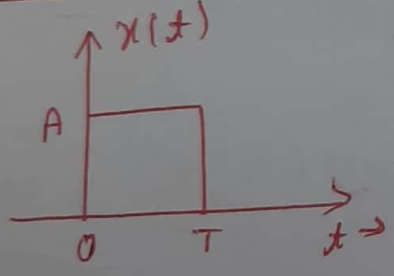
$$= \int_{-\infty}^{\infty} x(\lambda)\, e^{-jwt_0}\, e^{-jw\lambda}\, d\lambda$$

$$= e^{-jwt_0} \underline{\int_{-\infty}^{\infty} x(\lambda)\, e^{-jw\lambda}\, d\lambda}$$

$$= e^{-jwt_0}\, X(\omega)$$

(1) Find the FT of the signal.

(1)

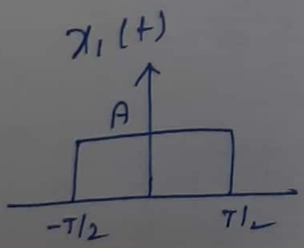


Let

$x_1(t) = A\,\mathrm{rect}\left(\frac{t}{T}\right) \xleftrightarrow{FT} AT\,\mathrm{sinc}(fT)$

or

$AT\,\mathrm{sinc}\left(\frac{\omega T}{2\pi}\right)$



The given signal

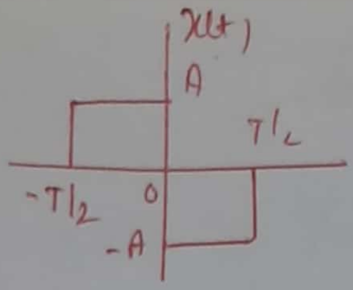
$x(t) = x_1(t - T/2)$

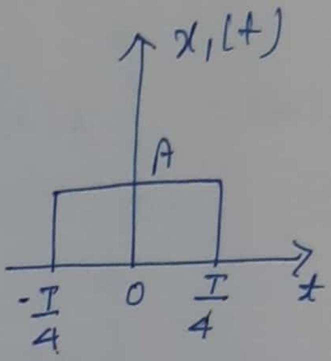
Taking FT

$X(w) = X_1(w)\, e^{-jwT/2}$

$\therefore\; X(w) = AT\,\mathrm{sinc}(fT) \cdot e^{-jwT/2}$

②





Let $x_1(t) = A\,\text{rect}\left(\frac{t}{T/2}\right)$

$$x_1(t) \xleftrightarrow{FT} \frac{AT}{2}\,\text{sinc}\left(f\frac{T}{2}\right) \text{ or } \frac{AT}{2}\,\text{sinc}\left(\frac{\omega T}{4\pi}\right)$$

given signal $x(t) = x_1(t+T/4) - x_1(t-T/4)$

Taking FT

$$X(\omega) = X_1(\omega)e^{\frac{j\omega T}{4}} - X_1(\omega)e^{-j\omega T/4}$$

$$= X_1(\omega)\left[\frac{e^{j\omega T/4} - e^{-j\omega T/4}}{2j}\right] \times 2j$$

$$= \frac{AT}{2}\,\text{sinc}\left(\frac{fT}{2}\right) 2j\, \underline{\sin(\omega T/4)} \rightarrow \sin\left(\frac{\pi f T}{2}\right)$$

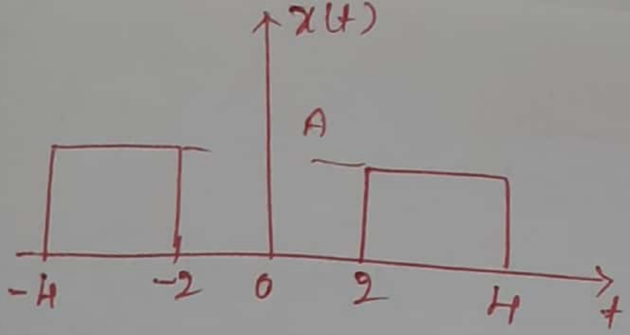
$$X(\omega) = \frac{AT}{2}\,\text{sinc}\left(\frac{\omega T}{4\pi}\right) \cdot 2j \sin\left(\frac{\omega T}{4}\right)$$

$$\boxed{X(\omega) = ATj\,\text{sinc}\left(\frac{\omega T}{4\pi}\right)\sin\left(\frac{\omega T}{4}\right)}$$

or

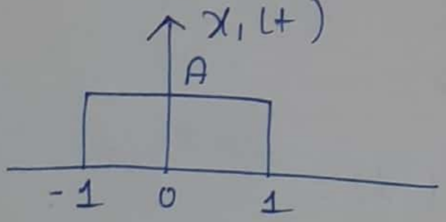
$$\boxed{X(\omega) : jAT\,\text{sinc}\left(\frac{fT}{2}\right) \cdot \sin\left(\frac{\pi f T}{2}\right)}$$

③



let

$x_1(t) = A\,\mathrm{rect}\left(\frac{t}{2}\right)\ (T=2)$



$x_1(t) \xleftrightarrow{FT} 2A\,\mathrm{sinc}(2f)$

given signal

$x(t) = x_1(t+3) + x_1(t-3)$

Taking FT

$$X(\omega) = X_1(\omega)e^{j\omega 3} + X_1(\omega)e^{-j\omega 3}$$

$$= 2A\,\mathrm{sinc}(2f)\,e^{j\omega 3} + 2A\,\mathrm{sinc}(2f)\,e^{-j\omega 3}$$

$$= 2A\,\mathrm{sinc}(2f)\left[\frac{e^{j\omega 3}+e^{-j\omega 3}}{2}\right]\times 2$$

$$\boxed{X(\omega) = 4A\,\mathrm{sinc}(f2)\cdot\cos(3\omega) \text{ or } 4A\,\mathrm{sinc}\left(\frac{\omega}{\pi}\right)\cos(3\omega)}$$

④ $\delta(t+t_0)$

WKT $\delta(t) \xleftrightarrow{FT} 1$

$\delta(t+t_0) \xleftrightarrow{FT} 1\cdot e^{j2\pi f t_0}$

⑤ $\delta(t-t_0)$

$\delta(t-t_0) \xleftrightarrow{FT} 1\cdot e^{-j2\pi f t_0}$

③ Frequency shift property :-

If $x(t) \xleftrightarrow{FT} X(\omega)$

then

$$y(t) = x(t)\, e^{j\omega_0 t} \xleftrightarrow{FT} X(\omega - \omega_0)$$

multiplying by an exponential in time-domain will result in shift in freq domain

Proof :-

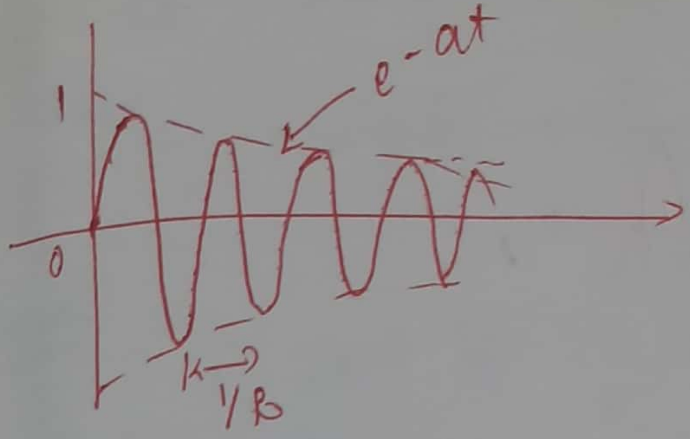
$$Y(\omega) \triangleq \int_{-\infty}^{+\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{j\omega_0 t} \cdot e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega - \omega_0) t}\, dt$$

$$= X(\omega - \omega_0)$$

find the Fourier transforms of the foll signal



exponential damped sinusoidal

$e^{-at}u(t).\sin w_0t$

Let $x_1(t) = e^{-at}u(t)$

$$X_1(\omega) = \frac{1}{a+j\omega}$$

given signal $x(t) = x_1(t).\sin w_0 t$

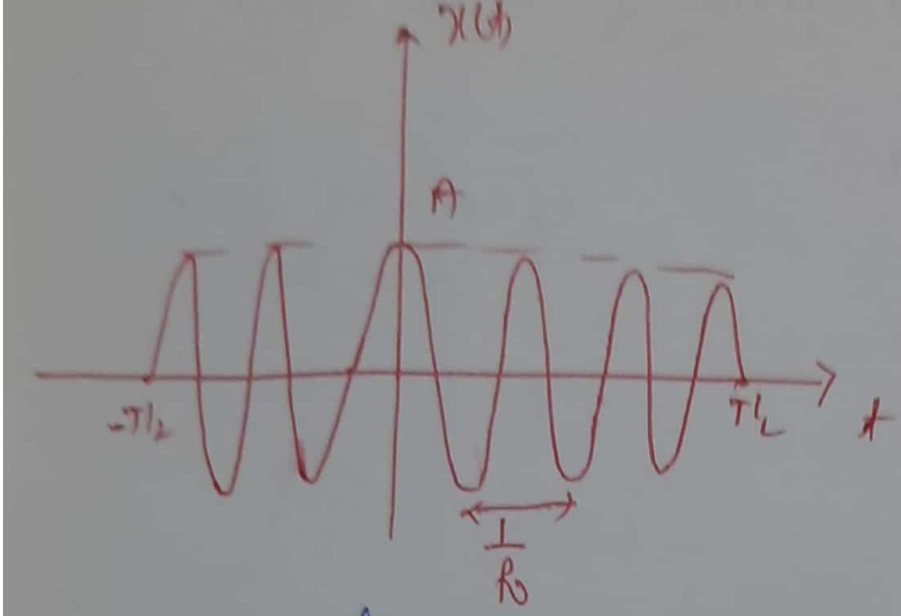
$$= \frac{x_1(t)}{2j}\left[e^{jw_0t} - e^{-jw_0t}\right]$$

Applying shifting prop

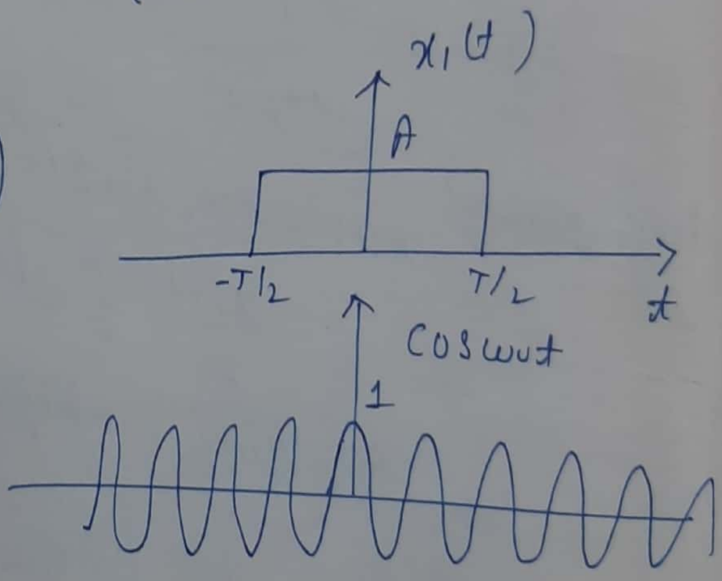
~~$X(\omega) = \frac{1}{2j}\left[X_1j(\right.$~~

$$X(\omega) = \frac{1}{2j}\left[X_1(j(\omega-w_0)) - X_1(j(\omega+w_0))\right]$$

$$= \frac{1}{2j}\left[\frac{1}{a+j(\omega-w_0)} - \frac{1}{a+j(\omega+w_0)}\right]$$



$$x(t) = \begin{cases} A\cos w_0 t \,, & -T/_2 \le t \le T/_2 \\ 0 \,, & e.w \end{cases}$$



Let $x_1(t) = A\,rect\left(\frac{t}{T}\right)$

$X_1(\omega) = AT\,sinc(fT)$

or

$AT\,sinc\left(\frac{\omega T}{2\pi}\right)$

$= x(t)$

Given signal can be written as $x(t) = x_1(t) \cdot \cos w_0 t$

$$= \frac{x_1(t)}{2}\left[e^{j w_0 t} + e^{-j w_0 t}\right]$$

$$X(\omega) = \frac{1}{2}\left[X_1(j(\omega - \omega_0)) + X_1(j(\omega + \omega_0))\right]$$

$$X(\omega) = \frac{AT}{2}\,sinc\left[\frac{(\omega - \omega_0)T}{2\pi}\right] + \frac{AT}{2}\,sinc\left(\frac{(\omega + \omega_0)T}{2\pi}\right)$$

⑧ differentiation in freq domain :

If $x(t) \xleftrightarrow{FT} X(\omega)$

then $-jt\, x(t) \xleftrightarrow{FT} \dfrac{d X(\omega)}{d\omega}$

proof :

by defn of FT,

$$X(\omega) \triangleq \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

differentiating w.r.t 'w'

$$\frac{d X(\omega)}{d\omega} = \int_{-\infty}^{\infty} x(t).\, [(-jt) e^{-j\omega t}]\, dt$$

$$= \int_{-\infty}^{\infty} [-jt\, x(t)]\, e^{-j\omega t}\, dt$$

i.e, $F\{-jt\, x(t)\} = \dfrac{d X(\omega)}{d\omega}$

$$\therefore \boxed{-jt\, x(t) \xleftrightarrow{FT} \frac{d X(\omega)}{d\omega}}$$

or

$$t\, x(t) \xleftrightarrow{FT} +j \frac{d X(\omega)}{d\omega}$$

$\boxed{-\frac{1}{j} = j}$

⑥ differentiation in time - domain

If $x(t) \xleftrightarrow{FT} X(\omega)$

Then $\frac{d\,x(t)}{dt} \xleftrightarrow{FT} j\omega\, X(\omega)$

Proof:- from the defn of inverse FT,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega .$$

differentiating w.r.t to 't'

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\left(j\omega\, e^{j\omega t}\right) d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} [j\omega\, X(\omega)]\, e^{j\omega t}\, d\omega$$

From the above eqn, we can write

$$F^{-1}\{j\omega X(\omega)\} = \frac{d\,x(t)}{dt}$$

or

$$\boxed{\frac{dx(t)}{dt} \xleftrightarrow{FT} j\omega X(\omega)}$$

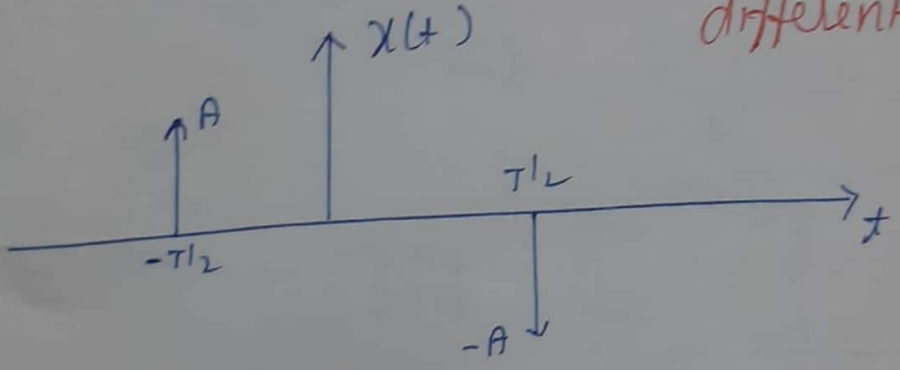
|||'y $\frac{d^2x(t)}{dt^2} \xleftrightarrow{FT} (j\omega)^2 X(\omega)$

in general

$$\frac{d^n x(t)}{dt^n} \xleftrightarrow{FT} (j\omega)^n X(\omega)$$

① Find FT of the foll signau

$x(t) = A[\delta(t+T/2) - \delta(t-T/2)]$ using differentiation Property



I $x(t) = A[\delta(t+T/2) - \delta(t-T/2)]$

WKT $\delta(t) \xleftrightarrow{FT} 1$

$\therefore \delta(t \pm T/2) \xleftrightarrow{FT} e^{\pm 2\pi f T/2} = e^{\pm j\pi f T}$

$X(\omega) = A\left[\frac{e^{j\pi f T} - e^{-j\pi f T}}{2j}\right] \times 2j$

$= A2j \sin\pi f T$

$= j2\pi f AT \, \text{sinc}(fT)$

Let us .

WKT ~~x~~.

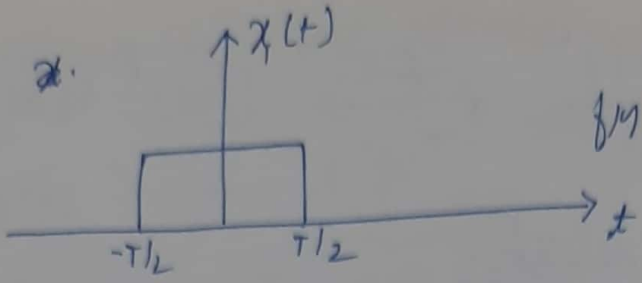


fig (a).

Note: unit impulse is considered as 1st derivative of ~~continuous time~~ unit step

$$\frac{d x(t)}{dt} = \delta(t)$$

$$\frac{d\, u(t+T/2)}{dt} = \delta(t+T/2)$$

$$\frac{d\, u(t-T/2)}{dt} = \delta(t-T/2)$$

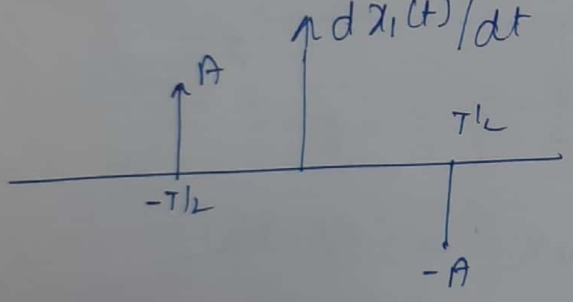


fig (b)

$x_1(t) = A\, rec\left(\frac{t}{T}\right)$

differentiating w.r.t 't' we get signal shown in fig (b)

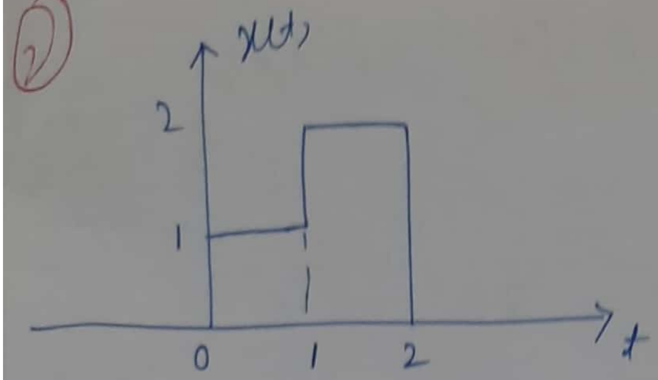
$$x(t) = \frac{d\, x_1(t)}{dt}$$

using differentiating property

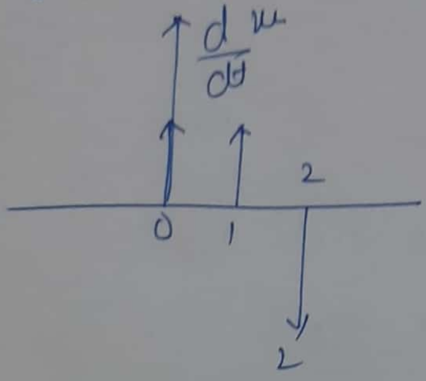
$$X(w) = (jw)\, X_1(jw)$$

$$= (jw)\, AT\, sinc(fT)$$

②



In this method we keep on differentiating until we get a impulse fn



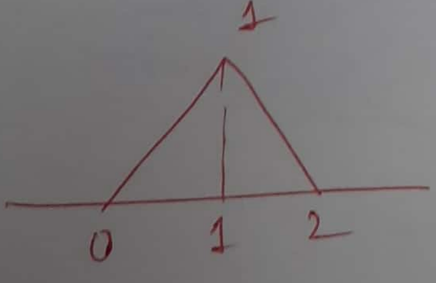
$$\frac{d\,x(t)}{dt} = \delta(t) + \delta(t-1) - 2\delta(t-2)$$

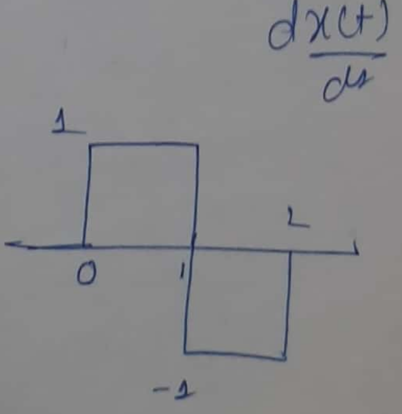
taking FT on both side

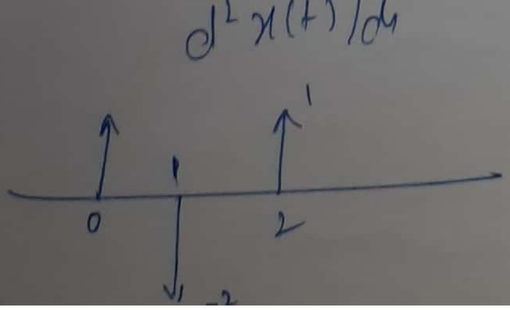
$$j\omega X(\omega) = 1 + e^{-j\omega} - 2\times e^{-j2\omega}$$

$$\therefore X(\omega) = \frac{1}{j\omega}\left[1 + e^{-j\omega} - 2e^{-2j\omega}\right]$$

③







Xerox - # 03

end.

(14 pgs)

$$\frac{d^2x(t)}{dt^2} = \delta(t) - 2\delta(t-1) + \delta(t-2)$$

Taking FT

$$j^2w^2 X(w) = 1 - 2e^{-jw} + e^{-2jw}$$

$$\therefore X(w) = \frac{1}{j^2w^2}\left[1 - 2e^{-jw} + e^{-j2w}\right]$$

④ $t e^{-at} u(t)$

$$x(t) = t e^{-at} u(t) = t \cdot x_1(t)$$

$$t \cdot x_1(t) \xleftrightarrow{FT} j \frac{d X_1(jw)}{dw}$$

$$x_1(t) = e^{-at} u(t) \xleftrightarrow{FT} X_1(w) = \frac{1}{a+jw}$$

$$\therefore X(w) = j \frac{d}{dw}\left(\frac{1}{a+jw}\right)$$

$$= j\left[\frac{-j}{(a+jw)^2}\right] = \frac{-j^2}{(a+jw)^2}$$

$$= \frac{1}{(a+jw)^2}$$

Integration (or) Accumulation

If $x(t) \xleftrightarrow{FT} X(\omega)$

Then

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau \longleftrightarrow \frac{X(\omega)}{j\omega} ~~+ \pi X(0)\,\delta(\omega)~~$$

$$+ \pi X(j0)\,\delta(\omega)$$

Proof Let $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

differentiating w.r.t t

$$\frac{dy(t)}{dt} = x(t)$$

Taking FT using differentiating property

$$(j\omega)\, Y(\omega) = X(\omega).$$

or $$\boxed{Y(\omega) = \frac{X(\omega)}{j\omega}}$$

The above reln applies only to the signal whose avg value is zero. i.e. $X(0) = 0$

* in general.

$$\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{FT} \frac{X(\omega)}{j\omega} + \pi X(0)\,\delta(\omega)$$

where $X(0) = X(j\omega)\big/_{\omega = 0}$

3.94 Prb

in OK

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⑧ Convolution property;

If $x(t) \xleftrightarrow{FT} X(j\omega)$

& $y(t) \xleftrightarrow{FT} Y(j\omega)$

Then $z(t) = x(t) * y(t) \xleftrightarrow{FT} X(j\omega)\, Y(j\omega) = Z(j\omega)$

Proof:

$$Z(j\omega) \triangleq \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} [x(t)*y(t)]\, e^{-j\omega t}\, dt$$

wkt $x(t)*y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$

$$= \int_{t=-\infty}^{\infty} \left[\int_{\tau=-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau\right] e^{-j\omega t}\, dt$$

$$= \int_{\tau=-\infty}^{\infty} x(\tau) \left[\int_{t=-\infty}^{\infty} y(t-\tau)\, e^{-j\omega t}\, dt\right] d\tau$$

Let $t-\tau = \lambda$ then $dt = d\lambda$

$$= \int_{\tau=-\infty}^{\infty} x(\tau) \int_{\lambda=-\infty}^{\infty} y(\lambda)\, e^{-j\omega(\tau+\lambda)}\, d\lambda\, d\tau$$

$$= \int_{\tau=-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau} d\tau \int_{\lambda=-\infty}^{\infty} y(\lambda)\, e^{-j\omega\lambda} d\lambda$$

$$= X(j\omega)\, Y(j\omega)$$

⑨ Modulation property.

If $x(t) \xleftrightarrow{FT} X(j\omega)$

$y(t) \xleftrightarrow{FT} Y(j\omega)$

then $z(t) = x(t)\, y(t) \xleftrightarrow{FT} Z(\omega) = \frac{1}{2\pi}[X(j\omega) * Y(j\omega)]$

Proof

$$Z(j\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(t)\, y(t)\, e^{-j\omega t} dt$$

From the defn of inverse FT, we have,

$$x(t) = \frac{1}{2\pi} \int_{\lambda=-\infty}^{\infty} X(\lambda)\, e^{j\lambda t} d\lambda$$

$$Z(j\omega) = \int_{t=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{\lambda=-\infty}^{\infty} X(\lambda)\, e^{j\lambda t} d\lambda\right] y(t)\, e^{-j\omega t} dt$$

$$= \frac{1}{2\pi} \int_{\lambda=-\infty}^{\infty} x(\lambda) \int_{t=-\infty}^{\infty} y(t)\, e^{-j(\omega-\lambda)dt}\, d\lambda$$

$$= \frac{1}{2\pi} \int_{\lambda=-\infty}^{\infty} x(\lambda)\; Y(\omega-\lambda)\, d\lambda$$

$$= \frac{1}{2\pi} \left[X(\omega) * Y(\omega)\right]$$

⑩ symmetry property

If $x(t) \xleftrightarrow{FT} X(\omega)$

Then (a) If $x(t)$ is real, then $X^*(j\omega) = X(-j\omega)$

(b) If $x(t)$ is imaginary, then $X^*(\omega) = -X(-\omega)$

Proof:

$$X(\omega) \triangleq \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt$$

Taking conjugate on both sides

$$X^*(\omega) = \int_{-\infty}^{\infty} x^*(t)\, e^{j\omega t}\, dt \longrightarrow ①$$

ⓐ if $x(t)$ is real then $x^*(t) = x(t)$

$$\therefore X^*(\omega) = \int_{-\infty}^{\omega} x(t)\, e^{+j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j(-\omega t)}\, dt = X(-\omega)$$

from def^n of Imaginary, then

$x^*(t) = -x(t)$

$(t) = -x(t)$

$x(t) =$

$$X^*(\omega) = \int_{-\infty}^{\infty} -x(t)\, e^{-j(-\omega t)} dt$$

$$= -X(-\omega)$$

⑪ Duality Property :-

If $\quad x(t) \xleftrightarrow{FT} X(j\omega)$

Then $\quad X(jt) \xleftrightarrow{FT} 2\pi\, x(-\omega)$

Proof - by def^n of inverse FT

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega$$

$$2\pi\, x(t) = \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega$$

changing t to $-t$

$$2\pi\, x(-t) = \int_{-\infty}^{\infty} X(j\omega)\, e^{-j\omega t} d\omega$$

interchanging 't' & 'ω'

$$2\pi\, x(-\omega) = \int_{-\infty}^{\infty} X(jt)\, e^{-j\omega t} dt \quad \text{—①}$$

comp eq ①

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$$

$$X(jt) \longleftrightarrow 2\pi\, x(-\omega)$$

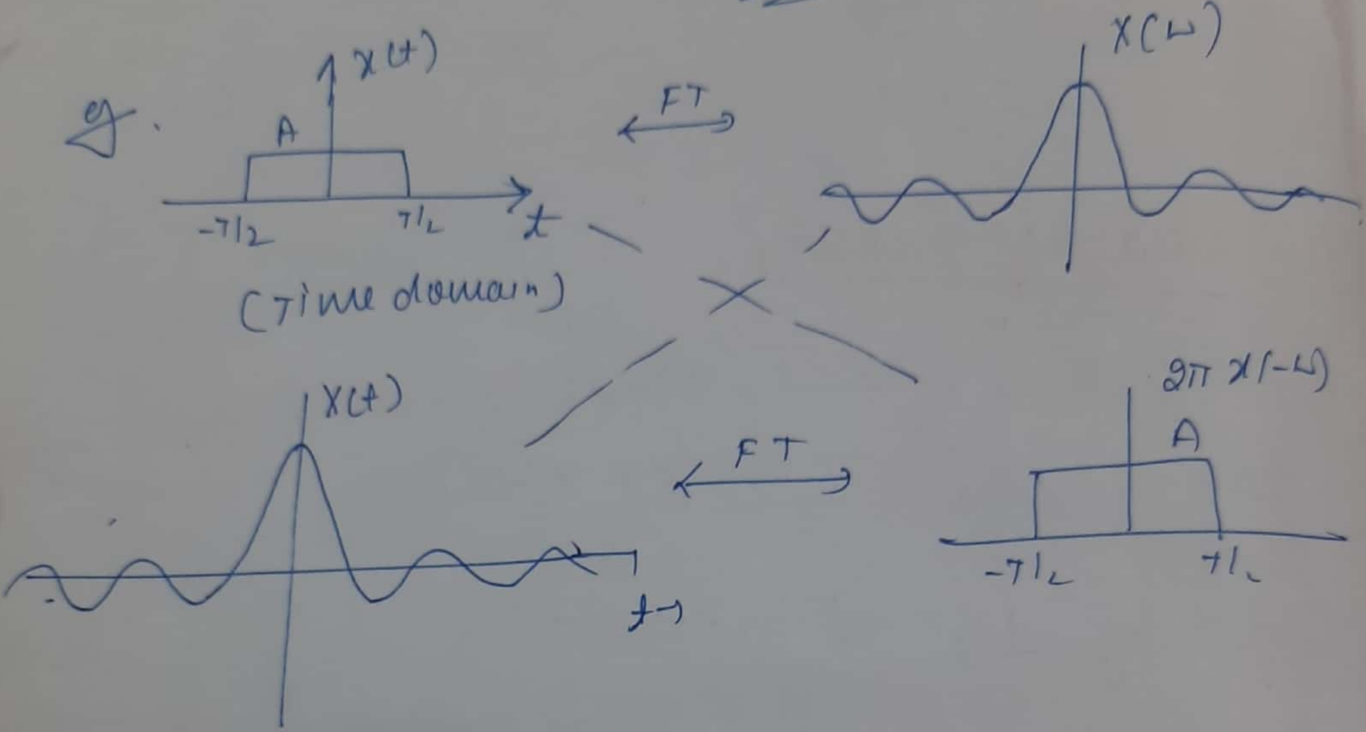
$u \quad F\{X(jt)\} = 2\pi\, x(-\omega)(\omega \ldots \quad d\lambda$ from

$2\pi\, x(-\omega)$

or

$\therefore \quad X(jt) \longleftrightarrow 2\pi\, x(-\omega)$

eg.



⑫ parseval's theorem

If $\quad x(t) \xleftrightarrow{FT} X(\omega)$

then $\quad E = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega$

Proof:

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

$$= \int_{t=-\infty}^{\infty} x(t)\, x^*(t)\, dt \longrightarrow ①$$

from defn of IFT

$$x(t) = \frac{1}{2\pi}\int_{w=-\infty}^{\infty} X(w)\, e^{jwt}\, dw$$

Taking conjugates on both the sides

$$x^*(t) = \frac{1}{2\pi}\int_{w=-\infty}^{\infty} X^*(w)\, e^{-jwt}\, dw \longrightarrow ②$$

substituting eq ② in eq ①

$$E = \int_{t=-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{w=-\infty}^{\infty} X^*(w)\, e^{-jwt}\, dw\right] dt$$

$$= \frac{1}{2\pi}\int_{w=-\infty}^{\infty} X^*(w)\left[\int_{t=-\infty}^{\infty} x(t)\, e^{-jwt}\, dt\right] dw$$

$$= \frac{1}{2\pi}\int_{w=-\infty}^{\infty} X^*(w)\, X(w)\, dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(w)|^2\, dw$$

Problems;

(1) Find the FT of the foll signals using duality property

(1) $x(t) = A \cos 2\pi f_0 t$

$= \frac{A}{2}\left[e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right]$

WKT $\delta(t) \xleftrightarrow{FT} 1$

using duality property

$1 \xleftrightarrow{FT} \delta(f)$

or

$1 \xleftrightarrow{FT} 2\pi\,\delta(\omega)$

$\delta(t \pm t_0) \xleftrightarrow{FT} e^{\pm j2\pi f_0 t} \cdot 1$

$e^{\pm j2\pi f_0 t} \xleftrightarrow{Ft} \delta(f \mp f_0)$

$F\{x(t)\} = \frac{A}{2}\left\{\delta(f-f_0) + \delta(f+f_0)\right] = X(f)$

$= A\pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right] = X(j\omega)$

(2) ~~$x(t) =$~~ $\frac{1}{a^2+t^2}$ using FT pa

③ pr 3.31 → pg 3.58

scaling

If $x(t) \xleftrightarrow{FT} X(\omega)$

then $y(t) : x(at) \xleftrightarrow{FT} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

proof: $Y(\omega) \triangleq$

$$= \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$$

Let $a > 0$,

put $at = \lambda$

$\therefore dt = \frac{d\lambda}{a}$

$$Y(\omega) = \int_{-\infty}^{\infty} x(\lambda)\, e^{-j\omega \frac{\lambda}{a}}\, \frac{d\lambda}{a}$$

$$= \frac{1}{a} \int$$

$$= \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

The above integral is valid both $a > 0$ & $a < 0$ hence $|a|$ is used

Special/ -

$$x(at) \overset{FT}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

if a:-1

$$x(-t) \overset{FT}{\longleftrightarrow} \frac{1}{|(-1)|} X\left(\frac{\omega}{(-1)}\right)$$

$$\boxed{x(-t) \overset{FT}{\longleftrightarrow} X(-\omega)}$$

folding property,

Find inverse FTs

① $x(\omega) = \frac{j\omega}{(2+j\omega)^2}$

WkT $\quad e^{-2t}u(t) \xleftrightarrow{FT} \frac{1}{2+j\omega}$

using freq differentiation property

$$-j\cdot t\cdot e^{-2t}u(t) \longleftrightarrow \frac{d}{dt}\left[\frac{1}{2j\omega}\right)$$

$$-j\cdot t e^{-2t}u(t) \longleftrightarrow \frac{-j}{(2+j\omega)\ 2}$$

$$t e^{-2t}u(t) \longleftrightarrow \frac{1}{(2+j\omega)^2}$$

using time diff' propr

$$\frac{d}{dt}\left[t\cdot e^{-2t}u(t)\right] \xleftrightarrow{FT} j\omega\,\frac{1}{2+j\omega)^2}$$

$$\therefore x(t) = \frac{d}{dt}\left[t\cdot e^{-2t}u(t)\right]$$

$$= \left[-2t\,e^{-2t} + e^{-2t}\right]u(t)$$

$$= (1-2t)\,e^{-2t}u(t)$$

Xerox-2(end) 7 Pgs.

②
$$X(j\omega) = \frac{5j\omega+12}{(j\omega)^2+5j\omega+6}$$

$$= \frac{5j\omega+12}{(j\omega+3)(j\omega+2)}$$

$$= \frac{A}{(j\omega+3)} + \frac{B}{(j\omega+2)}$$

$$= \frac{3}{j\omega+3} + \frac{2}{j\omega+2}$$

$$e^{-at}u(t) \xleftrightarrow{F.T} \frac{1}{a+j\omega}$$

$$x(t) = 3e^{-3t}u(t) + 2e^{-2t}u(t)$$

③
$$X(j\omega) = \frac{-j\omega}{(j\omega)^2+3j\omega+2}$$

$$= \frac{-j\omega}{(j\omega+1)(j\omega+2)}$$

$$= \frac{1}{(j\omega+1)} - \frac{2}{j\omega+2}$$

$$x(t) = \left(e^{-t} - 2e^{-2t}\right)u(t)$$

Example 3.95 Find the Fourier transform of,

$$x(t) = \frac{2}{t^2 + 1}$$

Solution. From duality property we have,

$$\text{If} \qquad x(t) \overset{\text{FT}}{\longleftrightarrow} X(j\omega)$$

$$\text{then} \qquad X(jt) \overset{\text{FT}}{\longleftrightarrow} 2\pi x(-\omega)$$

We know that,

$$e^{-a|t|} \overset{\text{FT}}{\longleftrightarrow} \frac{2a}{a^2 + \omega^2}$$

$$\therefore \quad x(t) = e^{-|t|} \overset{\text{FT}}{\longleftrightarrow} X(j\omega) = \frac{2}{1 + \omega^2}$$

∴ Using duality property we have,

$$X(jt) = \frac{2}{t^2+1} \overset{\text{FT}}{\longleftrightarrow} 2\pi x(-\omega) = 2\pi e^{-|-\omega|} = 2\pi e^{-|\omega|}$$

$$\therefore \quad \frac{2}{t^2+1} \overset{\text{FT}}{\longleftrightarrow} 2\pi e^{-|\omega|}$$

Example 3.96 Find the Fourier transform of the signal $x(t)$ using appropriate properties.

$$x(\text{t}) = \frac{\text{d}}{\text{dt}}\left[\text{t}e^{-2\text{t}}\sin(\text{t})u(\text{t})\right]$$

Solution. Given :

$$\begin{aligned} x(\text{t}) &= \frac{\text{d}}{\text{dt}}[\text{t}e^{-2\text{t}}\sin(\text{t})u(\text{t})] \\ &= \frac{\text{d}}{\text{dt}}\left[\text{t}e^{-2\text{t}}\left(\frac{e^{j\text{t}}-e^{-j\text{t}}}{2j}\right)u(\text{t})\right] \\ &= \frac{1}{2j}\frac{\text{d}}{\text{dt}}\left[\text{t}e^{j\text{t}}e^{-2\text{t}}u(\text{t}) - \text{t}e^{-j\text{t}}e^{-2\text{t}}u(\text{t})\right] \end{aligned}$$

We have,

$$e^{-2\text{t}}u(\text{t}) \overset{\text{FT}}{\longleftrightarrow} \frac{1}{2+j\omega}$$

Using frequency differentiation property we get,

$$\begin{aligned} \text{t}e^{-2\text{t}}u(\text{t}) \overset{\text{FT}}{\longleftrightarrow} j\frac{\text{d}}{\text{d}\omega}\left[\frac{1}{2+j\omega}\right] \\ = \frac{1}{(2+j\omega)^2} \end{aligned}$$

Using frequency shift property we get,

$$\text{t}e^{j\text{t}}e^{-2\text{t}}u(\text{t}) \overset{\text{FT}}{\longleftrightarrow} \frac{1}{[2+j(\omega-1)]^2}$$

Similarly,

$$\text{t}e^{-j\text{t}}e^{-2\text{t}}u(\text{t}) \overset{\text{FT}}{\longleftrightarrow} \frac{1}{[2+j(\omega+1)]^2}$$

$$\therefore \quad \left[\text{t}e^{j\text{t}}e^{-2\text{t}}u(\text{t}) - \text{t}e^{-j\text{t}}e^{-2\text{t}}u(\text{t})\right] \overset{\text{FT}}{\longleftrightarrow} \left[\frac{1}{[2+j(\omega-1)]^2} - \frac{1}{[2+j(\omega+1)]^2}\right]$$

Using time differentiation property we get,

$$\frac{d}{dt}\left[te^{-2t}\sin(t)u(t)\right] \xleftrightarrow{FT} j\omega\cdot\frac{1}{2j}\left[\frac{1}{[2+j(\omega-1)]^2}+\frac{1}{[2+j(\omega+1)]^2}\right]$$

$$=\frac{\omega}{2}\left[\frac{1}{[2+j(\omega-1)]^2}-\frac{1}{[2+j(\omega+1)]^2}\right]$$

Example 3.97 Find the FT of the signal given by,

$$x(t)=\int_{-\infty}^{t}\frac{\sin(\pi\tau)}{\pi\tau}d\tau$$

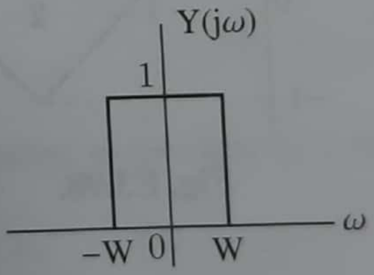
Solution. From integration property we have,

$$\text{If} \qquad x(t) \xleftrightarrow{FT} X(j\omega)$$

$$\text{then} \qquad \int_{-\infty}^{t}x(\tau)d\tau \xleftrightarrow{FT} \frac{X(j\omega)}{j\omega}+\pi X(j0)\delta(\omega) \qquad \text{(E3.97.1)}$$

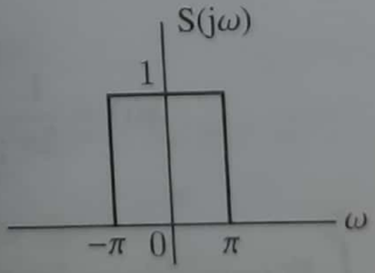
We know that,

$$y(t)=\frac{\sin(Wt)}{\pi t} \xleftrightarrow{FT}$$



(Refer to example 3.76)

$$\therefore \quad s(t)=\frac{\sin(\pi t)}{\pi t} \xleftrightarrow{FT}$$



Using eqn. (E3.97.1) we get,

$$x(t)=\int_{-\infty}^{t}s(\tau)d\tau=\int_{-\infty}^{t}\frac{\sin(\pi\tau)}{\pi\tau}d\tau \xleftrightarrow{FT} X(j\omega)=\pi \qquad ;\ \omega=0$$

$$=\frac{1}{j\omega} \qquad ;\ -\pi<\omega<\pi$$

$$= \qquad ;\ \text{otherwise}$$

10.5.17

⇒ Fourier representation of signals (MODULE-4)

Fourier series ⟶ Periodic signals — CTFS / FS (first fourier series)
— DTFS

Fourier transformation ⟶ aperiodic signal — CTFT / FT
— DTFT.

⇒ Fourier transform representation

(i) Discrete time fourier transform (DTFT) representation of aperiodic signal.

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega n}).\underbrace{e^{j\Omega n}}_{\text{synthesis}}\, d\Omega \longrightarrow \text{IDTFT}$$

$$\text{where } \underbrace{\boxed{X(e^{j\Omega n}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}}}_{\text{Analysis}} \longrightarrow \text{DTFT} \quad \ldots\ldots\ldots (1)$$

NOTE $X(e^{j\Omega n})$ is also represented as $X(\Omega)$ or $X(j\Omega)$

↓
X is a function of $e^{j\Omega n}$ or Ω

If we replace $e^{j\Omega}$ by z we get $\sum_{n=-\infty}^{\infty} x(n) z^{-n} \rightarrow$ Z transform.

All properties of z-transforms holds good for fourier transform by replacing Z by $e^{j\Omega}$. NOTE $x[n] \xleftrightarrow{FT} X(\Omega)$

→ Properties of DTFT

(i) Linearity property

$$\underbrace{a\,x[n] + b\,y[n]}_{z(n)} \xleftrightarrow{DTFT} \underbrace{a\,X(\Omega) + b\,Y(\Omega)}_{Z(\Omega)}$$

Proof: $Z(\Omega) = \sum_{n=-\infty}^{\infty} z(n)\, e^{-j\Omega n}$.

$$Z(\Omega) = \sum_{n=-\infty}^{\infty} (a\,x[n] + b\,y[n])\, e^{-j\Omega n}$$

$$Z(\Omega) = a \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n} + b \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\Omega n}$$

$$Z(\Omega) = a\,X(\Omega) + b\,Y(\Omega).$$

$$\therefore \boxed{a\,x(n) + b\,y(n) \xleftrightarrow{DTFT} a\,X(\Omega) + b\,Y(\Omega)}$$

(ii) Time shifting property

If $x[n] \xleftrightarrow{DTFT} X(\Omega)$

then $x(n-n_0) \xleftrightarrow{DTFT} \underset{(z^{-n_0})}{e^{-j\Omega n_0}}\, X(\Omega)$.

(iii) Frequency shift property

If $x(n) \xleftrightarrow{DTFT} X(\Omega)$

$$x(n)\, e^{j\Omega\beta} \xleftrightarrow{DTFT} X(\Omega - \beta)$$

(iv) Differentiation in frequency domain

If $X[n] \xleftrightarrow{DTFT} X(\Omega)$

Then $-jn\, x(n] \xleftrightarrow{DTFT} \dfrac{d\, X(\Omega)}{d\Omega}$.

(v) **convolution property :**

If $x(n) \xleftrightarrow{DTFT} X(\Omega)$

$y(n) \xleftrightarrow{DTFT} Y(\Omega)$

then $x(n] * y(n) \xleftrightarrow{DTFT} X(\Omega).Y(\Omega)$.

Let $z(n) = x(n) * y(n]$

$Z(\Omega) = X(\Omega).Y(\Omega)$

Proof : $Z(\Omega) = \sum_{n=-\infty}^{\infty} z[n]\, e^{-j\Omega n}$ .... (1)

$$Z(\Omega) = \sum_{n=-\infty}^{\infty} (x(n) * y(n]) \, e^{-j\Omega n}$$

$$Z(\Omega) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x(k)\, y(n-k)\, e^{-j\Omega n} \text{ .... (2)}$$

$$Z(\Omega) = \sum_{k=-\infty}^{\infty} x(k) \sum_{l=-\infty}^{\infty} y(l)\, e^{-j\Omega(k+l)}$$

$\because \left(\begin{array}{l} n-k = l \\ n = k+l \end{array} \right)$

$$Z(\Omega) = \sum_{k=-\infty}^{\infty} x(k)\, e^{-j\Omega k} \sum_{l=-\infty}^{\infty} y(l)\, e^{-j\Omega l}$$

$Z(\Omega) = X(\Omega).\, Y(\Omega)$

$$\therefore \boxed{x(n) * y(n] \xleftrightarrow{DTFT} X(\Omega) \cdot Y(\Omega)}$$

(vi) **Modulation / multiplication** (multiplication in time domain is convolution in frequency domain it is reverse of above propert

If $x(n) \xleftrightarrow{DTFT} X(\Omega)$

$y(n) \xleftrightarrow{DTFT} Y(\Omega)$

then $x[n] . y[n] \xleftrightarrow{FT} \frac{1}{2\pi}\left(X(\Omega) * y(\Omega)\right)$

(vii) Parseval's theorem

If $x(n) \xleftrightarrow{DTFT} X(\Omega)$

then Energy $= \sum_{n=-\infty}^{\infty} |x(n)|^2 \xleftrightarrow{DTFT} \frac{1}{2\pi} \int_{2\pi} |X(\Omega)|^2 \, d\Omega.$

$\int_{2\pi} = \int_{-\pi}^{\pi}$ or $\int_{0}^{2\pi}$.

Proof: $E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \ldots\ldots\ldots\ldots (1)$

WKT $|x(n)|^2 = x(n) . x^*(n) \ldots\ldots\ldots (2)$

$\downarrow$ conjugate of $x(n)$.

From IDTFT, $x(n) = \frac{1}{2\pi} \int_{2\pi} X(\Omega) . e^{j\Omega n} \, d\Omega \ldots\ldots (3)$

Applying conjugate on both sides

$x^*(n) = \frac{1}{2\pi} \int_{2\pi} X^*(\Omega) \, e^{-j\Omega n} \, d\Omega \ldots\ldots (4)$

Simplifying $x^*(n)$ in equation (1) (eq^n (2), (4) in eq^n (1)).

$$E = \sum_{n=-\infty}^{\infty} x(n) . \frac{1}{2\pi} \int_{2\pi} X^*(\Omega) \, e^{-j\Omega n} \, d\Omega$$

$$E = \frac{1}{2\pi} \int_{2\pi} X^*(\Omega) \, d\Omega \left[\sum_{n=-\infty}^{\infty} x(n) . e^{-j\Omega n} \right]$$

$$E = \frac{1}{2\pi} \int_{2\pi} X^*(\Omega) \, X(\Omega) \, d\Omega.$$

$$\boxed{E = \frac{1}{2\pi} \int_{2\pi} |X(\Omega)|^2 \, d\Omega}$$

viii)

Conjugate property

If $x(n) \xleftrightarrow{DTFT} X(\Omega)$

Then $x^*(n) \xleftrightarrow{DTFT} X^*(-\Omega)$

Let $y(n) = x^*(n)$, $Y(\Omega) = X^*(-\Omega)$.

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} y(n) \, e^{-j\Omega n}$$

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} x^*(n) \, e^{-j\Omega n}$$

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} [x(n) \cdot e^{j\Omega n}]^*$$

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} [x(n) \cdot e^{-j(-\Omega) n}]^*$$

$$Y(\Omega) = X^*(-\Omega)$$

$$\therefore \boxed{x^*(n) \xleftrightarrow{DTFT} X^*(-\Omega)}$$

Properties of DTFT

(1) Linearity.

$$x[n] \longleftrightarrow X(\Omega)$$

$$y[n] \longleftrightarrow Y(\Omega)$$

$$z[n] = a x[n] + b y[n] \longleftrightarrow aX(\Omega) + bY(\Omega) = Z(\Omega)$$

proof

$$Z(\Omega) = \sum_{n=-\infty}^{\infty} z(n)\, e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(a x[n] + b y[n]\right) e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} a x[n] e^{-j\Omega n} + \sum_{n=-\infty}^{\infty} b y[n] e^{-j\Omega n}$$

$$= a X(\Omega) + b Y(\Omega)$$

(2) Time Shift or Translation

$$x[n] \longleftrightarrow X(\Omega)$$

$$x[n-n_0] = y[n] \longleftrightarrow X(\Omega)\, e^{-j\Omega n_0} = Y(\Omega)$$

proof:

$$Y(\Omega) = \left\{ \sum_{n=-\infty}^{\infty} x[n-n_0]\, e^{-j\Omega n} \right.$$

$$\boxed{e^{-j\Omega n_0} X(\Omega)}$$

let $n - n_0 = \lambda$, $n = \lambda + n_0$

$$Y(\Omega) = \sum_{\lambda=-\infty}^{\infty} x(\lambda)\, e^{-j\Omega(\lambda+n_0)}$$

$$= e^{-j\Omega n_0} \sum_{\lambda=-\infty}^{\infty} x(\lambda)\, e^{-j\Omega\lambda}$$

3) Frequency shift or frequency Translation.

If $x(n) \xleftrightarrow{DTFT} X(\Omega)$

$y(n) = x(n) e^{j\beta n} \longleftrightarrow X(\Omega - \beta) = Y(\Omega)$

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{j\beta n} e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) e^{-j(\Omega-\beta)n}$$

$$= X(\Omega-\beta)$$

(4) ~~Scaling~~ Frequency differentiation

If $x(n) \xleftrightarrow{DTFT} X(\Omega)$

then $-jn\, x(n) \xleftrightarrow{DTFT} \frac{dX(\Omega)}{d\Omega}$

Proof:
$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$$

$$\frac{dX(\Omega)}{d\Omega} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n} \cdot (-jn)$$

$$= \sum_{n=-\infty}^{\infty} [x(n) -jn]\, e^{-j\Omega n}$$

$dX(\Omega) \xleftrightarrow{DTFT} x(n)(-jn)$

⑤ Convolution (TD)

⑧

if $x(n) \xleftrightarrow{DTFT} X(\Omega)$

$y(n) \xleftrightarrow{DTFT} Y(\Omega)$

then $z(n) = x(n) * y(n) \xleftrightarrow{DTFT} Z(\Omega) = X(\Omega)Y(\Omega)$

proof:

$$Z(\Omega) = \left\{ \sum_{n=-\infty}^{\infty} x(n) * y(n)\, e^{-j\Omega n} \right.$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x(k)\, y(n-k)\, e^{-j\Omega n}$$

let $n-k = l \Rightarrow n = l+k$

$$= \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x(k)\, y(l)\, e^{-j\Omega(l+k)}$$

$$= \sum_{l=-\infty}^{\infty} y(l) e^{-j\Omega l} \sum_{k=-\infty}^{\infty} x(k)\, e^{-j\Omega k}$$

$$Z(\Omega) = X(\Omega)\, Y(\Omega)$$

⑥ Multiplication or Modulation

if $x[n] \xleftrightarrow{DTFT} X(\Omega)$

$y[n] \xleftrightarrow{DTFT} Y(\Omega)$

$$z(n) = x(n)y(n) \xleftrightarrow{DTFT} \frac{1}{2\pi}\left[X(\Omega) * Y(\Omega)\right]$$

proof: $$Z(\Omega) = \sum_{n=-\infty}^{\infty} z(n)\, e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, y(n)\, e^{-j\Omega n} \longrightarrow ①$$

IDFT $$x(n) = \frac{1}{2\pi} \int_{2\pi} X(\beta)\, e^{j\beta n}\, d\beta \longrightarrow ②$$

Simplifying in eqn ①

$$Z(\Omega) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{2\pi} X(\beta)\, e^{j\beta n}\, d\beta \quad y(n)\, e^{-j\Omega n}$$

$$= \frac{1}{2\pi} \int_{2\pi} X(\beta) \sum_{n=-\infty}^{\infty} y(n)\, e^{-j(\Omega-\beta)n}\, d\beta$$

$$= \frac{1}{2\pi} \int_{2\pi} X(\beta)\; Y(\Omega-\beta)\; d\beta$$

$$Z(\Omega) = \frac{1}{2\pi} \quad X(\Omega) \circledast Y(\Omega)$$

⑨ Parseval's theorem

If $x[n] \xleftrightarrow{DTFT} X(\Omega)$

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(\Omega)|^2 \, d\Omega$$

proof:

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, x^*(n) \rightarrow ①$$

WKT $x(n) = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\, e^{j\Omega n} \, d\Omega$

$$x^*(n) = \frac{1}{2\pi}\int_{2\pi} X^*(\Omega)\, e^{-j\Omega n} \, d\Omega \rightarrow ②$$

Simplifying in eqn ②

$$= \sum_{n=-\infty}^{\infty} x(n) \frac{1}{2\pi}\int_{2\pi} X^*(\Omega)\, e^{-j\Omega n} \, d\Omega$$

$$= \frac{1}{2\pi}\int_{2\pi} X^*(\Omega)\, d\Omega \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$= \frac{1}{2\pi}\int_{2\pi} X^*(\Omega)\, X(\Omega) \, d\Omega$$

$$= \frac{1}{2\pi}\int_{2\pi} |X(\Omega)|^2 \, d\Omega \quad //$$

⑧ conjugate property

⑧

If $x[n] \xleftrightarrow{DTFT} X(\Omega)$

$y(n) = x^*(n) \xleftrightarrow{DTFT} X^*(-\Omega)$

proof:

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x^*(n)\, e^{-j\Omega n}$$

$$= \left[\sum_{n=\infty}^{\infty} x(n)\, e^{j\Omega n}\right]^*$$

$$= \left[\sum_{n=-\infty}^{\infty} x(n)\, e^{-j(-\Omega)n}\right]^*$$

$$Y(\Omega) = X^*(-\Omega)$$

Properties of DTFT - Problems.

1) $x(n) = a^n u(n) \qquad a<1$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

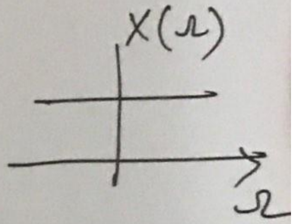
$$= \sum_{n=0}^{\infty} a^n \cdot e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} (a e^{-j\Omega})^n = \frac{1}{1-ae^{-j\Omega}}.$$

2) $x(n) = \delta(n)$

$$X(\Omega) = \sum_{n=\infty}^{\infty} x(n)\, e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} \delta(n)\, e^{-j\Omega n}$$

$$= 1\, e^{-j\Omega n} \;@\; n=0 = 1.$$



(3) $x(n) = 1 \qquad |n| \le M$

$\quad\;\; = 0 \qquad |n| > M$

$$X(\Omega) = \sum_{n=\infty}^{\infty} x(n)\, e^{-j\Omega n} = \sum_{n=-M}^{M} 1 \cdot e^{-j\Omega n}$$

Let $n+M = l \Rightarrow n = l-M$

$$X(\Omega) = \sum_{l=0}^{2M} 1\, e^{-j\Omega(l-M)}$$

$$= e^{+j\Omega M} \sum_{l=0}^{2M} e^{-j\Omega l}$$

$$= e^{j\Omega M}\, \frac{1-e^{-j\Omega(2M+1)}}{1-e^{-j\Omega}}$$

4) $x(n) = (-1)^n u(n)$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot 1\, e^{-j\Omega n} = \sum_{n=0}^{\infty} (-1\, e^{-j\Omega})^n$$

$$= \frac{1}{1+e^{-j\Omega}}$$

$$X(\Omega) = \frac{e^{j\Omega/2}}{2\cos\Omega/2}.$$

5) $x(n) = u(n) - u(n-6)$

$$X(\Omega) = \sum_{n=0}^{5} x(n)\, e^{-j\Omega n}$$

$$= \sum_{n=0}^{5} e^{-j\Omega n} = \frac{1-e^{-j6\Omega}}{1-e^{-j\Omega}}$$

6) $x(n) = 2^n u(-n)$

$$X(\Omega) = \sum_{n=-\infty}^{0} 2^n \cdot e^{-j\Omega n}$$

let $m = -n$

$$X(\Omega) = \sum_{m=0}^{\infty} 2^{-m} e^{j\Omega m}$$

$$= \sum_{m=0}^{\infty} \left(\tfrac{1}{2} e^{j\Omega}\right)^n$$

$$= \frac{1}{1-\tfrac{1}{2} e^{j\Omega}}$$

7.) $x(n) = \left(\frac{1}{4}\right) u(n+4)$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$= \sum_{n=-4}^{+\infty} \left(\frac{1}{4}\right)^{n} e^{-j\Omega n}$$

$$= \sum_{n=-4}^{\infty} \left(\frac{1}{4} e^{-j\Omega}\right)^{n} \qquad \sum_{n=k}^{\infty} a^{n} = \frac{a^{k}}{1-a}$$

$$= \frac{\left(\frac{1}{4} e^{j\Omega}\right)^{-4}}{1 - \frac{1}{4} e^{-j\Omega}}$$

$$X(\Omega) = \frac{256\, e^{4j\Omega}}{1 - \frac{1}{4} e^{-j\Omega}}$$

⑧ $x(n) = a^{n} \sin \Omega_0 n\, u(n) \qquad |a| < 1$

$$X(\Omega) = \sum_{n=0}^{\infty} a^{n} \sin \Omega_0 n\, e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} a^{n} \left[\frac{e^{j\Omega_0 n} - e^{-j\Omega_0 n}}{2j}\right] e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} a^{n} e^{(j\Omega_0 - j\Omega)n} - \sum_{n=0}^{\infty} a^{n} e^{-j\Omega_0 n} . e^{-j\Omega n}$$

$$= $$

$$X(\Omega) = \frac{a(\sin \Omega_0) e^{-j\Omega}}{1 - (2a \cos \Omega_0) e^{-j\Omega} + a^{2} e^{-j2\Omega}}.$$

9) $x(n) = \left(\frac{1}{2}\right)^{n} \left[u(n+3) - u(n-2)\right]$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$= \sum_{n=-3}^{2} \left(\frac{1}{2}\right)^{n} e^{-j\Omega n}$$

$$\sum_{n=N_1}^{N_2} a^{n} =$$

let $l = n+3$

$$X(\Omega) = \sum_{l=0}^{4} \left(\frac{1}{2} e^{-j\Omega}\right)^{l-3}$$

$$= \left(\frac{1}{2}\right)^{-3} e^{+3j\Omega} \sum_{l=0}^{4} \left(\frac{1}{2} e^{-j\Omega}\right)^{l}$$

$$X(\Omega) = 8 e^{j3\Omega} \left[\frac{1 - \left(\frac{1}{2}\right)^{5} e^{-j\Omega 5}}{1 - \frac{1}{2} e^{-j\Omega}}\right]$$

10) $x(n) = n \left(\frac{1}{2}\right)^{|n|}$

Sol^{n}

$$a^{|n|} \xleftrightarrow{\text{DTFT}} \frac{1-a^{2}}{1-2a\cos\Omega + a^{2}} \quad |a|<1$$

$$\left(\frac{1}{2}\right)^{|n|} \xleftrightarrow{\text{DTFT}} \frac{1-\left(\frac{1}{2}\right)^{2}}{1-2\left(\frac{1}{2}\right)\cos\Omega + \left(\frac{1}{2}\right)^{2}}$$

$$= \frac{3/4}{5/4 - \cos\Omega}$$

roblems on DTFT.

1. Find the DTFT of

(a) $x(n) = \{1, 2, \underset{\uparrow}{3}, 2, 1\}$ evaluate $X(\Omega)$ @ $\Omega = 0$.

Sol^n

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$$

$$= \sum_{n=-2}^{2} x(n) e^{-j\Omega n}$$

$$= x(-2)e^{+2j\Omega} + x(-1)e^{j\Omega} + x(0) + x(1)e^{-j\Omega} + x(2)e^{-j2\Omega}.$$

$$= e^{2j\Omega} + 2e^{j\Omega} + 3 + 2e^{-j\Omega} + e^{-2j\Omega}$$

$$X(\Omega) = 2\cos 2\Omega + 3 + 4\cos\Omega$$

$$X(\Omega)\Big|_{\Omega=0} = 9.$$

(b) $x(n) = (0.5)^{n+2} u[n]$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} (0.5)^n (0.5)^2 e^{-j\Omega n}$$

$$= \frac{1}{4} \sum_{n=0}^{\infty} \left[(0.5 e^{-j\Omega})\right]^n$$

$$= \frac{1}{4} \frac{1}{1 - 0.5 e^{-j\Omega}}$$

substrat
gate

②

$$x(n) = n\,(0.5)^{2n}\,u(n)$$

$$\sum n a^{n} = \frac{a}{(1-a)^2}$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} n(0.5)^{2n} e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} n \cdot \left((0.5)^2 e^{-j\Omega}\right)^{n}$$

$$= \sum_{n=0}^{\infty} \left(\tfrac{1}{4} e^{-j\Omega}\right)^{n}$$

$$= \frac{\tfrac{1}{4} e^{-j\Omega}}{\left(1-\tfrac{1}{4} e^{-j\Omega}\right)^2}$$

④ $x(n) = -a^{n} u(-n-1)$

$u($

$-n-1 \geq 0$

$-n \geq 1$

$n \leq -1$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{-1} -a^{n} e^{-j\Omega n}$$

$$= -\sum_{m=1}^{\infty} a^{-m} e^{+j\Omega m}$$

$$= -\left[\sum_{m=}^{\infty} a^{-m} e^{j\Omega n} + 1 - 1\right]$$

$$= 1 - \sum_{m=0}^{\infty} \left(a^{-1} e^{j\Omega}\right)^{n}$$

$$= 1 - \frac{1}{1-a^{-1}e^{j\Omega}} = \frac{\cancel{1} - a^{-1}e^{j\Omega} - \cancel{1}}{1-a^{-1}e^{j\Omega}} = \underline{\frac{1}{1-ae^{-j\Omega}}}$$

$x(n) = \left(\frac{1}{2}\right)^{n} u[n-4]$ ④

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

$$= \sum_{n=4}^{\infty} \left(\frac{1}{2}\right)^{n} e^{-j\Omega n}$$

let $n+4 = l \Rightarrow n = l+4$

$$= \sum_{l=0}^{\infty} \left(\frac{1}{2}\right)^{l+4} e^{-j\Omega(l+4)}$$

$$= \left(\frac{1}{2}\right)^{4} e^{-j\Omega 4} \sum_{l=0}^{\infty} \left(\frac{1}{2} e^{-j\Omega}\right)^{l}$$

$$= \left(\frac{1}{2}\right)^{4} e^{-j\Omega 4} \left[\frac{1}{1-\frac{1}{2}e^{-j\Omega}}\right]$$

$=$

$$|X(\Omega)| = \left(\frac{1}{2}\right)^{4} \frac{1}{\sqrt{\left(1-\frac{1}{2}\cos\Omega\right)^{2} + \left(\frac{1}{2}\sin\Omega\right)^{2}}}$$

$$|X(\Omega)| = -4\Omega - \tan^{-1}\left(\frac{\frac{1}{2}\sin\Omega}{1-\frac{1}{2}\cos\Omega}\right)$$

$$e^{\pm j\theta} = 1\angle\pm\theta$$

⑤

$x(n) = a^{|n|} \quad a<1$

$x(n) = a^{-n}u[-n-1] + a^{n}u[n]$

$\downarrow \qquad\qquad n>0$

$n<0$

$$X(\Omega) = \sum_{m=-\infty}^{-1} a^{-n} e^{-j\Omega n} + \sum_{n=0}^{\infty} a^{n}u[n]e^{-j\Omega n}$$

$$= \sum_{m=1}^{\infty} a^{m} e^{j\Omega m}$$

$$= 1-1+\sum_{m=1}^{\infty} a^{m} e^{j\Omega m n}$$

$$= -1 + \sum_{m=0}^{\infty} a^{m} e^{j\Omega m}$$

$$= -1 + \frac{1}{1-ae^{j\Omega}} + \frac{1}{1-ae^{-j\Omega}}$$

$$= \frac{ae^{j\Omega}}{1-ae^{j\Omega}} + \frac{1}{1-ae^{-j\Omega}}$$

$$= \frac{1-a^{2}}{1+a^{2}-2a\cos\Omega} \qquad |a|<1$$

(h) $x[n] = u[n]$

$$\sum_{n=-\infty}^{\infty} u[n] \not< \infty$$

$$\delta(n) = u[n] - u[n-1]$$

$$\delta[n] = x[n] - x[n-1]$$

$$1 = X(\Omega) - X(\Omega) e^{-j\Omega}$$

$$1 = X(\Omega)\left[1 - e^{-j\Omega}\right]$$

$$X(\Omega) = \frac{1}{1 - e^{-j\Omega}} + A\,\delta(\Omega) \quad @ \ \Omega = 0.$$

(i) $x(n) = \delta(6 - 3n)$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$= e^{-j\Omega n}\Big|_{6 - 3n = 0}$$

$$= e^{-j\Omega n}\Big|_{n=2} = e^{-j2\Omega} \,//$$

(j) Find IDFT of $X(\Omega) = 2\pi\,\delta(\Omega - \Omega_0)$

$$x(n) = \frac{1}{2\pi} \int_{\pi}^{\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega$$

$$x(n) = \frac{1}{2\pi} \int_{\pi}^{\pi} 2\pi\,\delta(\Omega - \Omega_0)\, e^{j\Omega n}\, d\Omega$$

5.5.2017

1) Find DTFT of $x(n) = \delta[n]$.

Solⁿ: $$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} \delta[n] . e^{-j\Omega n}$$

$$X(\Omega) = 1 . e^{-j\Omega n} \quad \text{at } n = 0$$

$$X(\Omega) = 1 . e^{-j\Omega(0)}$$

$$\underline{\underline{X(\Omega) = 1}}$$

2) $x[n] = a^n . u[n]$ where $a < 1$

Sol $$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} a^n\, u[n] . e^{-j\Omega n}$$

$$X(\Omega) = \sum_{n=0}^{\infty} a^n . 1 . e^{-j\Omega n}$$ $\{\because u(n) = 1$ from $0 < n < \infty\}$

$$X(\Omega) = \sum_{n=0}^{\infty} \left(a . e^{-j\Omega}\right)^n$$

WKT $$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

$$\underline{X(\Omega) = \frac{1}{1 - ae^{-j\Omega}}}$$

(3) $x(n) = \{1, 2, \underset{\uparrow}{3}, 2, 1\}$

Sol^n: $x(-2) = 1$

$x(-1) = 2$

$x(0) = 3$

$x(1) = 2$

$x(2) = 1$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$$

$$X(\Omega) = \sum_{n=-2}^{2} x(n) e^{-j\Omega n}$$

$$X(\Omega) = (1) e^{-j\Omega(-2)} + 2 e^{-j\Omega(-1)} + 3 e^{-j\Omega(0)} + 2 e^{-j\Omega(1)} + 1 e^{-j}$$

$$X(\Omega) = e^{2j\Omega} + 2e^{j\Omega} + 3 + 2e^{-j\Omega} + e^{-2j\Omega}$$

$$\underline{\underline{X(\Omega) = 2(e^{j\Omega} + e^{-j\Omega}) + (e^{2j\Omega} + e^{-2j\Omega}) + 3}}$$

(4) $x(n) = (-1)^n u[n].$

Sol^n: $$X(\Omega) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\Omega n}$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} (-1)^n u(n) e^{-j\Omega n}$$

$$X(\Omega) = \sum^{\infty} (-1 \cdot e^{-j\Omega})^n$$

$$X(\Omega) = \frac{1}{1 - \left(-e^{-j\Omega}\right)}$$

$$X(\Omega) = \frac{1}{1+e^{-j\Omega}}$$

(5.) $x(n) = 2^n . u[-n]$

Sol^n: $$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} 2^n . u[-n] . e^{-j\Omega n}$$

$$X(\Omega) = \sum_{n=-\infty}^{0} 2^n .(1). e^{-j\Omega n}$$ $\{\because u[-n]$ is 1 from $-\infty < n < 0$

Let $l = -n$.

$\therefore n = -\infty \quad l = \infty$

$n = 0 \quad l = 0.$

$$X(\Omega) = \sum_{l=\infty}^{0} 2^{-l} . e^{-j\Omega(-l)}$$

$$X(\Omega) = \sum_{l=0}^{\infty} 2^{-l} . e^{j\Omega l}$$

$$X(\Omega) = \sum_{l=0}^{\infty} \left(\frac{e^{j\Omega}}{2}\right)^l$$

$$X(\Omega) = \frac{1}{1 - \frac{e^{j\Omega}}{2}}$$ $\left\{\because \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}\right\}$

(6) $x(n) = \left(\frac{1}{4}\right)^n u(n+4]$.

Method - 1

Solⁿ $$X(\Omega) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\Omega n}$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{4}\right)^n u(n+4)\, e^{-j\Omega n}$$

$$X(\Omega) = \sum_{n=-4}^{\infty} \left(\frac{1}{4}\right)^n \cdot 1 \cdot e^{-j\Omega n}$$

$$X(\Omega) = \sum_{n=-4}^{\infty} \left(\frac{1}{4} \cdot e^{-j\Omega}\right)^n$$

WKT = $$\sum_{n=K}^{\infty} a^n = \frac{a^K}{1-a}$$

$$\therefore X(\Omega) = \frac{\left(\frac{e^{-j\Omega}}{4}\right)^{-4}}{1 - \left(\frac{e^{-j\Omega}}{4}\right)}$$

$$X(\Omega) = \frac{4^4 \cdot e^{4j\Omega}}{(4 - e^{-j\Omega}) \cdot \frac{1}{4}}$$

$$X(\Omega) = \frac{4^5 \cdot e^{4j\Omega}}{4 - e^{-j\Omega}}$$

Method - 2.

Using properties

If $x(n) \xleftrightarrow{DTFT} X(\Omega)$

Then $x(n-n_0) \xleftrightarrow{DTFT} e^{-j\Omega n_0} . X(\Omega)$. {Time shifting property}.

$$x(n) = \left(\frac{1}{4}\right)^n . u[n+4].$$

$$x(n) = 4^4 . \frac{1}{4^4} \cdot \left(\frac{1}{4}\right)^n u[n+4]$$

$$x[n] = 4^4 . \left(\frac{1}{4}\right)^{n+4} . u(n+4)$$

consider $x'(n) = \left(\frac{1}{4}\right)^n . u(n)$

$$X'(\Omega) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{4}\right)^n . u(n) . e^{-j\Omega n}$$

$$X'(\Omega) = \sum_{n=0}^{\infty} \left(\frac{1}{4} . e^{-j\Omega}\right)^n$$

$$X'(\Omega) = \frac{1}{1 - \frac{e^{-j\Omega}}{4}}$$

$$X'(\Omega) = \frac{4}{4 - e^{-j\Omega}}$$

$$\therefore X(\Omega) = e^{-j\Omega(-4)} . \frac{4}{4 - e^{-j\Omega}} . 4^4$$

$$X(\Omega) = \frac{4^5 . e^{4j\Omega}}{4 - e^{-j\Omega}}$$

(7) $x(n) = u(n) - u[n-6]$

Sol: $$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}.$$

Method 1:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} u(n)\, e^{-j\Omega n} - \sum_{n=-\infty}^{\infty} u(n-6)\, e^{-j\Omega n}$$ { linearity property

$$X(\Omega) = \sum_{n=0}^{\infty} e^{-j\Omega n} - \sum_{n=6}^{\infty} e^{-j\Omega n}$$

$$X(\Omega) = \frac{1}{1-e^{-j\Omega}} - \frac{\left(e^{-j\Omega}\right)^{6}}{1-e^{-j\Omega}}$$

$$X(\Omega) = \frac{1}{1-e^{-j\Omega}}\left(1 - e^{-6j\Omega}\right)$$

$$X(\Omega) = \frac{1-e^{-6j\Omega}}{1-e^{-j\Omega}}$$

Method 2:

$u(n) - u(n-6) = 1$ from $n = 0$ to 6.

$$\therefore X(\Omega) = \sum_{n=0}^{6} e^{-j\Omega n}.$$

$$X(\Omega) = \frac{1 - e^{-6j\Omega}}{1 - e^{-j\Omega}} \qquad \left\{ \sum_{n=0}^{N} a^n = \right\}$$

(8) $x(n) = \left(\frac{1}{2}\right)^n u[n-2].$

Sol^n: Using time shifting property

If $x(n) \xleftrightarrow{DTFT} X(\Omega)$

Then $x(n - n_0) \xleftrightarrow{DTFT} e^{-j\Omega n_0} X(\Omega).$

Given $x(n) = 2^{-2} \left(\frac{1}{2}\right)^n \cdot \left(\frac{1}{2}\right)^{-2} u(n-2)$

$$x(n) = \left(\frac{1}{2}\right)^{n-2} \cdot \frac{1}{4} \cdot u(n-2)$$

$$x(n) = \frac{1}{4} \cdot \left(\frac{1}{2}\right)^{n-2} \cdot u(n-2)$$

consider $x'(n) = \left(\frac{1}{2}\right)^n \cdot u(n)$

$$X'(\Omega) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u(n)\, e^{-j\Omega n}$$

$$X'(\Omega) = \sum_{n=0}^{\infty} \left(\frac{1}{2} \cdot e^{-j\Omega}\right)^n$$

$$X'(\Omega) = \frac{1}{1 - \frac{e^{-j\Omega}}{2}}$$

$$\therefore X(\Omega) = \frac{1}{4} \cdot e^{-j\Omega \cdot (2)} \cdot \frac{1\,(2)}{2 - e^{-j\Omega}}$$

$$X(\Omega) = \frac{1}{2} \cdot \frac{e^{-2j\Omega}}{2 - e^{-j\Omega}}$$

(9) $x(n) = a^n \sin \Omega_0 n . u(n)$

Sol: $X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$

WKT $\sin \Omega_0 n = \frac{e^{j\Omega_0 n} - e^{-j\Omega_0 n}}{2j}$

$\therefore x(n) = a^n \left(\frac{e^{j\Omega_0 n} - e^{-j\Omega_0 n}}{2j} \right) . u(n)$

$$x(n) = \frac{a^n . u(n) . e^{j\Omega_0 n}}{2j} - \frac{a^n . u(n) . e^{-j\Omega_0 n}}{2j}$$

$\therefore$ From frequency shift property,

If $x(n) \xleftrightarrow{DTFT} X(\Omega)$

then $x(n) . e^{jn\beta} \xleftrightarrow{DTFT.} X(\Omega - \beta)$ (doubt).

$\therefore$ Let $x'(n) = a^n u(n)$

$\therefore X'(\Omega) = \sum_{n=0}^{\infty} a^n e^{-j\Omega n}$

$$X'(\Omega) = \frac{1}{1 - ae^{-j\Omega}}.$$

$$\therefore X(\Omega) = \frac{1}{2j} \frac{1}{1 - a^{-j(\Omega-\Omega_0)}} - \frac{1}{2j} \frac{1}{1 - a^{-j(\Omega-(-\Omega_0))}}$$

$$X(\Omega) = \frac{1}{2j} \left\{ \frac{1}{1 - ae^{-j(\Omega-\Omega_0)}} - \frac{1}{1 - ae^{-j(\Omega+\Omega_0)}} \right\}$$

17.5.17

(10) $x[n] = (\sin \frac{\pi}{4} n)(\frac{1}{4})^n u(n-1).$

Sol^n: $(\frac{1}{4})^n \cdot u(n-1) = (\frac{1}{4})^n (\frac{1}{4})^{-1} \cdot (\frac{1}{4})^{+1} \cdot u(n-1)$

$= (\frac{1}{4})^{n-1} \cdot (\frac{1}{4})\ u(n-1).$

consider $(\frac{1}{4})^n \cdot u(n) = x_1(n).$

$$\therefore x_1(\Omega) = \sum_{n=-\infty}^{+\infty} (\tfrac{1}{4})^n \cdot u(n) \cdot e^{-j\Omega n}$$

$$x_1(\Omega) = \sum_{n=0}^{\infty} (\tfrac{1}{4} \cdot e^{-j\Omega})^n$$

$$x_1(\Omega) = \frac{1}{1 - \frac{e^{-j\Omega}}{4}}.$$

From time shifting property,

$$x_2(\Omega) = \tfrac{1}{4} \cdot (\tfrac{1}{4})^{n-1} u(n-1) \xleftrightarrow{DTFT} \tfrac{1}{4} \cdot e^{-j\Omega(1)} \cdot \frac{1}{1 - \frac{e^{-j\Omega}}{4}}$$

$$\therefore x_2(\Omega) = \frac{1}{4} \cdot e^{-j\Omega} \frac{4}{4 - e^{-j\Omega}} = \frac{e^{-j\Omega}}{4 - e^{-j\Omega}}$$

$$x(n) = \left(\sin \frac{\pi}{4} n\right) \left(\frac{1}{4}\right)^n u(n-1)$$

$$x(n) = \left(\frac{e^{j\frac{\pi}{4}n} - e^{-j\pi/4 n}}{2j}\right) \left(\frac{1}{4}\right) \left(\frac{1}{4}\right)^{n-1} \cdot u(n-1).$$

$$x(n) = \frac{e^{j\pi/4 n}}{2j} \cdot x_2(n) - \frac{1}{2j} e^{-j\pi/4 n} \cdot x_2(n) \quad \left\{ \text{where } x_2(n) = \frac{1}{4} \cdot \left(\frac{1}{4}\right)^{n-1} \cdot u(n-1) \right\}$$

Applying frequency shift property if $x(n) \xleftrightarrow{DTFT} X(\Omega)$

Then $x(n)\, e^{jn\beta} \xleftrightarrow{DTFT} X(\Omega - \beta)$

$$\therefore X(\Omega) = \frac{1}{2j} \frac{e^{-j(\Omega - \pi/4)}}{4 - e^{-j(\Omega - \pi/4)}} - \frac{1}{2j} \frac{e^{-j(\Omega + \pi/4)}}{4 - e^{-j(\Omega + \pi/4)}}$$

$$X(\Omega) = \frac{1}{2j} \left[\frac{e^{-j(\Omega - \pi/4)}}{4 - e^{-j(\Omega - \pi/4)}} - \frac{e^{-j(\Omega + \pi/4)}}{4 - e^{-j(\Omega + \pi/4)}} \right]$$

---

1) Let $x(n)$ be a sequence $\{3, 0, 1, 4\}$ with a DTFT $X(e^{j\Omega})$. Evaluate the following functions of $X(e^{j\Omega})$ without computing $X(e^{j\Omega})$.

a) $X(e^{j0})$

b) $X(e^{j\pi})$

c) $\int_{-\pi}^{\pi} X(e^{j\Omega})\, d\Omega$

d) $\int_{-\pi}^{\pi} |X(e^{j\Omega})|^2 \, d\Omega$

e) $\int_{-\pi}^{\pi} \left| \frac{dX(e^{j\Omega})}{d\Omega} \right|^2 d\Omega$

Sol^n:

$X(e^{j\Omega}) = X(\Omega).$

WKT $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}.$

Let $\Omega = 0$

$X(e^{j0}) = \sum_{n=0}^{3} x[n] e^{-j0n}.$

$X(e^{j0}) = \sum_{n=0}^{3} x(n)$

$\underline{X(e^{j0}) = 3+0+1+4 = 8}$

b) $X(e^{j\pi}) = \sum_{n=0}^{3} x(n) e^{-j\pi n}.$

$X(e^{j\pi}) = 3 + (0) e^{-j\pi(1)} + (1) e^{-j\pi(2)} + 4 e^{-j\pi(3)}.$

$X(e^{j\pi}) = 3 + e^{-2\pi j} + 4 e^{-3\pi j}.$

$X(e^{j\pi}) = 3 + [\cos(-2\pi) + j\sin(-2\pi)] + 4[\cos(-3\pi) + j\sin(-3\pi)]$

$X(e^{j\pi}) = 3 + (1+0) + 4(-1)$

$X(e^{j\pi}) = 0$

c) $\int_{-\pi}^{\pi} X(e^{j\Omega})\, d\Omega$

$$X(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega})\, e^{j\Omega n}\, d\Omega$$ (from Inverse fourier transform)

Let $n = 0$

$$X(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) \cdot e^{0}\, d\Omega$$

$$\therefore \int_{-\pi}^{\pi} X(e^{j\Omega})\, d\Omega = 2\pi(3) = 6\pi$$

---

d) $\int_{-\pi}^{\pi} |X(e^{j\Omega})|^2\, d\Omega$

From parseval's theorem,

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\Omega})|^2\, d\Omega$$

$$2\pi \sum_{n=0}^{3} [x(n)]^2 = \int_{-\pi}^{\pi} |X(e^{j\Omega})|^2\, d\Omega$$

$$\therefore \int_{-\pi}^{\pi} |X(e^{j\Omega})|^2\, d\Omega = 2\pi\left[x(0)^2 + (x(1))^2 + (x(2))^2 + (x(3))^2 \right]$$

$$\int_{-\pi}^{\pi} |X(e^{j\Omega})|^2\, d\Omega = 2\pi\left[9 + 0 + 1 + 16 \right]$$

$$\int_{-\pi}^{\pi} |X(e^{j\Omega})|^2\, d\Omega = 52\pi$$

c). $\int_{-\pi}^{\pi} \left| \frac{d X(e^{j\Omega})}{d\Omega} \right|^2 d\Omega.$

From differentiation property,

$$-jn\, x(n) \xleftrightarrow{DTFT} \frac{d X(\Omega)}{d\Omega}$$

$$2\pi \sum_{n=0}^{3} |-jn\ x(n)|^2 = \int_{-\pi}^{\pi} \left| \frac{d X(\Omega)}{d\Omega} \right|^2 d\Omega.$$

$$\therefore \int_{-\pi}^{\pi} \left| \frac{d\ X(\Omega)}{d\Omega} \right|^2 d\Omega = 2\pi \left[\sum_{n=0}^{3} n^2\ x(n)^2 \right]$$

$$\int_{-\pi}^{\pi} \left| \frac{d X(\Omega)}{d\Omega} \right|^2 d\Omega = 2\pi \left[0^2 . x(0)^2 + 1^2 . x(1)^2 + 2^2 . X(2)^2 + 3^2 X(3)^2 \right]$$

$$= \underline{\underline{146\pi}} \text{ (doubt)}$$

⑬ $x[n] = n\,(0.5)^n u(n)$

Sol^n

WKT $a^n u(n) \xleftrightarrow{DTFT} \dfrac{1}{1-ae^{-j\Omega}}$.

$\therefore\ (0.5)^n u(n) \xleftrightarrow{DTFT} \dfrac{1}{1-0.5e^{-j\Omega}}$

Using differentiation property.

$$-jn\,x[n] \xleftrightarrow{DTFT} \frac{d\,X(\Omega)}{d\Omega}$$

$$n\,x[n] \xleftrightarrow{DTFT} j\,\frac{d\,X(\Omega)}{d\Omega}$$

$$\therefore\ n(0.5)^n u[n] \xleftrightarrow{DTFT} j\,\frac{d}{d\Omega}\left(\frac{1}{1-0.5e^{-j\Omega}}\right)$$

$$= j\;\frac{\left[(1-0.5e^{-j\Omega})\,0 - 1(-0.5e^{-j\Omega}\cdot(-j)\right)}{(1-0.5e^{-j\Omega})^2}$$

$$\text{DTFT}\left\{ n(0.5)^n u(n) \right\}$$

$$= \frac{j\left(-j\,0.5e^{-j\Omega}\right)}{\left(1-0.5e^{-j\Omega}\right)^2}$$

$$= \frac{0.5e^{-j\Omega}}{\left(1-0.5e^{-j\Omega}\right)^2}$$

⑭ $x[n] = \left(\frac{1}{4}\right)^n \cancel{*} u[n] * \left(\frac{1}{3}\right)^n u[n]$

Sol^n

WKT $a^n u(n) \xleftrightarrow{DTFT} \frac{1}{1-a e^{-j\Omega}}$

$$\left(\frac{1}{4}\right)^n u(n) \xleftrightarrow{DTFT} \frac{1}{1-1/4 e^{-j\Omega}}$$

$$(1/3)^n u(n) \xleftrightarrow{DTFT} \frac{1}{1-1/3 e^{-j\Omega}}$$

WKT $x(n) * y(n) \xleftrightarrow{DTFT} X(\Omega) Y(\Omega)$

$$\therefore \left(\frac{1}{4}\right)^n u(n) * \left(\frac{1}{3}\right)^n u(n) \xleftrightarrow{DTFT} \frac{1}{(1-1/4 e^{-j\Omega})} \cdot \frac{1}{(1-\frac{1}{3} e^{-j\Omega})}$$

$$\therefore X(\Omega) = \frac{1}{1 - 1/3 e^{-j\Omega} - \frac{1}{4} e^{-j\Omega} + \frac{1}{12} e^{-2j\Omega}}$$

$$X(\Omega) = \frac{1}{1 - \frac{7}{12} e^{-j\Omega} + \frac{1}{12} e^{-2j\Omega}}$$

⑮ $x[n] = \left(\frac{1}{4}\right)^n u[n-4]$

Sol^n

wkt $a^n u(n) \xleftrightarrow{DTFT} \frac{1}{1-ae^{-j\Omega}}$

$$x(n) = \left(\frac{1}{4}\right)^n \left(\frac{1}{4}\right)^{-4} \left(\frac{1}{4}\right)^4 u(n-4)$$

$$= \left(\frac{1}{4}\right)^4 \left(\frac{1}{4}\right)^{n-4} \cdot u(n-4)$$

$$x(n) = \frac{1}{256} \left(\frac{1}{4}\right)^{n-4} u(n-4)$$

$$\left(\frac{1}{4}\right)^n u(n) \xleftrightarrow{DTFT} \frac{1}{1-\frac{1}{4}e^{-j\Omega}}$$

Using time shifting property

$$\boxed{x(n-n_0) \xleftrightarrow{DTFT} e^{-j\Omega n_0} \cdot X(\Omega)}$$

$$\left(\frac{1}{4}\right)^{n-4} u(n-4) \xleftrightarrow{DTFT} e^{-j\Omega 4} \; X(\Omega)$$

DTFT of $(1/4)^n u(n)$

$$\therefore X(\Omega) = \frac{1}{256} e^{-j\Omega 4} \frac{1}{1-1/4\, e^{-j\Omega}}$$

⑫ The DTFT of a real signal is $X(e^{j\Omega})$. Express DTFT of each of the following signals in terms of $X(e^{j\Omega})$

(a) $x[-n]$

(b) $x[n] * x[-n]$

(c) $(-1)^n x[n] = \cos n\pi \, x[n]$.

(d) $(1+\cos n\pi)\, x[n]$.

Sol^n (a). $x[-n]$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n} \longrightarrow ①$$

$$\text{DTFT}\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n) e^{-j\Omega n}$$

let $n = -m$

$$= \sum_{m=-\infty}^{\infty} x(m) e^{+j\Omega m}$$

$$= \sum_{m=-\infty}^{\infty} x(m) e^{-j(-\Omega)m}$$

$$\text{DTFT}\{x(-n)\} = X(e^{-j\Omega})$$

$$= X(-\Omega)$$

$$\boxed{\therefore x(-n) \xleftrightarrow{\text{DTFT}} X(-\Omega) = X(e^{-j\Omega})}$$

$$\boxed{\text{(b)} \quad x[n] * x[-n]}$$

WKT

$$x[n] * y[n] \xleftrightarrow{\text{DTFT}} X(\Omega)Y(\Omega)$$

$$x[n] * x[-n] \xleftrightarrow{\text{DTFT}} X(\Omega)\,X(-\Omega)$$

$$\boxed{\text{(c)} \quad (-1)^n x[n] = \cos n\pi \; x[n]}$$

(c)

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

$$\text{DTFT}\{(-1)^n x[n]\} = \sum_{n=-\infty}^{\infty} (-1)^n x[n] e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} \cos n\pi \quad x[n] \; e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} \frac{e^{+j\pi n} + e^{-j\pi n}}{2} \cdot x[n] \; e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} \frac{x[n]}{2} \left[e^{-j(\Omega-\pi)n} + e^{-j(\Omega+\pi)n} \right]$$

$$= \frac{1}{2} \left\{ \sum_{n=-\infty}^{\infty} x[n] e^{-j(\Omega-\pi)n} + \sum_{n=-\infty}^{\infty} x[n] e^{-j(\Omega+\pi)n} \right\}$$

$$= \frac{1}{2} \left[X(\Omega-\pi) + X(\Omega+\pi) \right]$$

(d) $(1+\cos n\pi)\, x[n]$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

$$\text{DTFT}\{(1+\cos n\pi)x[n]\} = \sum_{n=-\infty}^{\infty} (1+\cos n\pi)\, x[n]\, e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n} + \sum_{n=-\infty}^{\infty} \cos n\pi \; x(n) e^{-j\Omega n}$$

$$\text{DTFT}\{(1+\cos n\pi)\, x[n]\} = X(\Omega) + \frac{1}{2}\left\{X(\Omega-\pi + X(\Omega+\pi)\right\}$$

module -5

Z - Transforms

Z - Transform, properties of convergence & Z-Transforms, inverse Z-Transforms, Causality & Stability, Transform analysis of LTI system

Z - Transforms

- Z - Transforms of a discrete time signal is defined as,

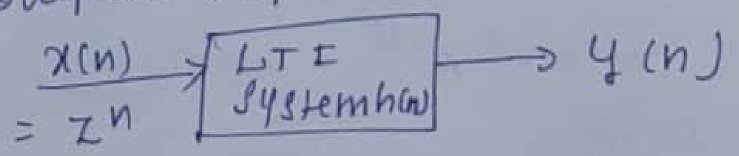
$$\boxed{\begin{array}{c} Z\{x(n)\} \triangleq X(z) \triangleq \sum_{n=-\infty}^{\infty} x(n) z^{-n} \\ \\ x(n) \xleftrightarrow{ZT} X(z) \end{array}}$$

Time domain Z - domain

Direct & Bilateral Z-Transform

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} \Longrightarrow \text{unilateral Z-transform}$$

consider a discrete LTI system with impulse response h(n) & $x(n) = z^n$ complex exponential i/p



$$y(n) = h(n) * x(n)$$

$$= \sum_{K=-\infty}^{\infty} h(K) \cdot x(n-k)$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, z^{n-k}$$

$$x(n) = z^n \quad \therefore \quad x(n-k) = z^{n-k}$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) \cdot z^n \cdot z^{-k}$$

$$y(n) = z^n \underbrace{\sum_{k=-\infty}^{\infty} h(k) \cdot z^{-k}}$$

$$= z^n \; H(z)$$

where $H(z) = \sum_{k=-\infty}^{\infty} h(k) \cdot z^{-k}$ — ①

(or) equivalently

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} \rightarrow ②$$

where $H(z)$ is known as Z-Transform of $h(n)$

* $Y(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$ — ③

here z = complex variable

$$z = re^{j\Omega}$$

$$= r\cos\Omega + jr\sin\Omega$$

$$= Re\{z\} + j\,Im\{z\} \longrightarrow ④$$

$r \rightarrow$ magnitude

$\Omega \rightarrow$ angles of 'z' respectively

substituting $z = re^{j\Omega}$ in eq ③

we get

$$X(re^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)\,(re^{j\Omega})^{-n}$$

$$'' \quad r^{-n}\,e^{-j\Omega n}$$

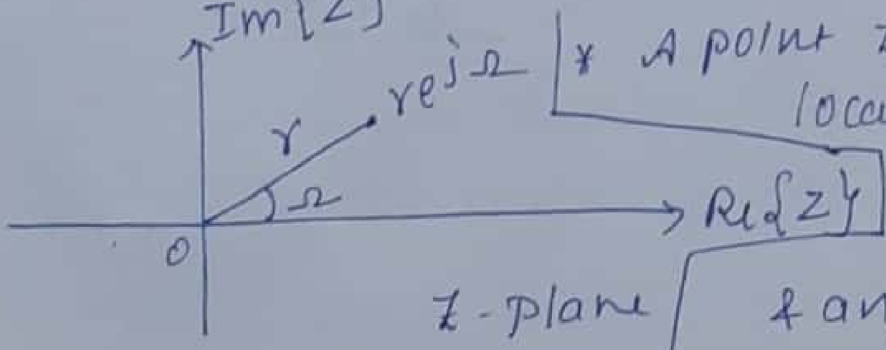
$$X(re^{j\Omega}) = \sum_{n=-\infty}^{\infty} \{x(n)\,r^{-n}\}\,e^{-j\Omega n} \longrightarrow ⑤$$

$$X(re^{j\Omega}) = \underset{\substack{\uparrow \\ DTFT}}{FT}\{x(n)\,r^{-n}) \rightarrow ⑥$$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)\,e^{-j\Omega n} \longrightarrow ⑦$$

if $r = 1$

$$X(e^{j\Omega}) = X(z)\Big|_{z=e^{j\Omega}} \longrightarrow ⑧$$



* A point $z = re^{j\Omega}$ is located at a distance 'r' from origin & angle Ω

Region of Convergence [ROC]

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= x(-\infty) z^{+\infty} + \cdots + x(-2) z^{+2} + x(-1) z^{1} + \cdots + x(0) z^{0} + x(1) z^{-1} + \cdots$$

$$X(z) = \sum_{n=-\infty}^{\infty} \{x(n) r^{-n}\} e^{-j\Omega n}$$

$$\sum_{n=-\infty}^{\infty} |x(n) r^{-n}| < \infty$$

The range of values of the complex-variable z for which Z-Transform converges is called Region of Convergence [ROC]

Z-Transforms & ROC of finite duration sequence

① Right sided sequence / Causal sequence / Positive time sequence

* A right sided sequence is one for which $x(n) = 0$ for all $n < n_0$, where n_0 is +ve or negative but finite

* If $n_0 \geq 0$, the resulting seq^n $x(n)$ is said to be either a causal seq^n or positive time seq^n

* For a causal finite sequence, the ROC is the entire Z-plane except for $Z = 0$

eg

$$x[n] = \{\underset{\uparrow}{1}, 2, 2, 1\}$$

WKT $X(Z) \triangleq \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$= \sum_{n=0}^{3} x(n) z^{-n}$$

$$= x(0) z^{-0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}$$

$$\boxed{X(z) = 1 + 2z^{-1} + 2z^{-2} + 1z^{-3}} \; ; \; \boxed{ROC: |z| > 0}$$

X(z) is finite for all values of z except at $z = \underline{0}$ $\therefore$ $\boxed{ROC : |z| > 0}$

$X(z) = 1 + 2 \cdot \frac{1}{z} + 2\frac{1}{z^2} + 1 \cdot \frac{1}{z^3}$ if $z = 0$ Then $\frac{1}{0} = \infty$

Left sided sequence or anticausal seqn

- A left sided seqn is one for which $x(n)=0$ for all $n > n_0$, where n_0 = +ve or -ve but finite. If $n_0 \leq 0$, the resulting seqn $x(n)$ is called anticausal.
- ROC is entire Z-plane excluding $Z = \infty$

eg $x(n) = \{1, 3, 5, \underset{\uparrow}{7}\}$

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-3}^{0} x(n) z^{-n}$$

$$= x(-3) z^3 + x(-2) z^2 + x(-1) z^1 + x(0) z^0$$

$$\boxed{X(z) = z^3 + 3z^2 + 5z + 7 \;;} \quad \boxed{ROC: |z| < \infty}$$

∴ $X(z)$ is finite for all values of z except for $z = \infty$ ∴ ROC: $|z| < \infty$

Double-sided sequence

- A signal that has finite duration in both the left & right sides is known as double-sided seqn
- ROC is entire Z-plane except at $Z = 0$ & $Z = \infty$

eg $x(n) = \{-1, 3, 5, \underset{\uparrow}{7}, 2, 5, 6\}$

$$X(Z) \triangleq \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-3}^{3} x(n) z^{-n}$$

$$\boxed{X(z) = -1z^{3} + 3z^{2} + 5z^{1} + 7 + 2z^{-1} + 5z^{-2} + 6z^{-3}}$$

ROC: entire z-plane except at $Z=0$ & $Z=\infty$

$$\boxed{\text{ROC}: 0 < |Z| < \infty}$$

Z-Transform & ROC of Infinite duration Seqn

① Positive time exponential sequence

$x(n) = a^{n} u(n)$

$$X(Z) \triangleq \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^{n} u(n) z^{-n}$$

WKT $u(n) = \begin{cases} 1 ; & n \geq 0 \\ 0 ; & n < 0 \end{cases}$

$$X(z) = \sum_{n=\infty}^{\infty} a^{n} . u(n) z^{-n} \text{ becomes}$$

$$= \sum_{n=0}^{\infty} a^{n} z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^{n}$$

$$\boxed{\text{WKT} \quad \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad ; \quad |a| < 1}$$

$$a = az^{-1}$$

$$X(z) = \frac{1}{1-(az^{-1})} = \frac{1}{1-\frac{a}{z}}$$

$$= \frac{1}{\frac{z-a}{z}}$$

$$\boxed{X(z) = \frac{z}{z-a}}$$

$$\boxed{\text{ROC: } |z| > |a|}$$

ROC :-

$$|a| < 1$$

$$|az^{-1}| < 1$$

$$\left|\frac{a}{z}\right| < 1$$

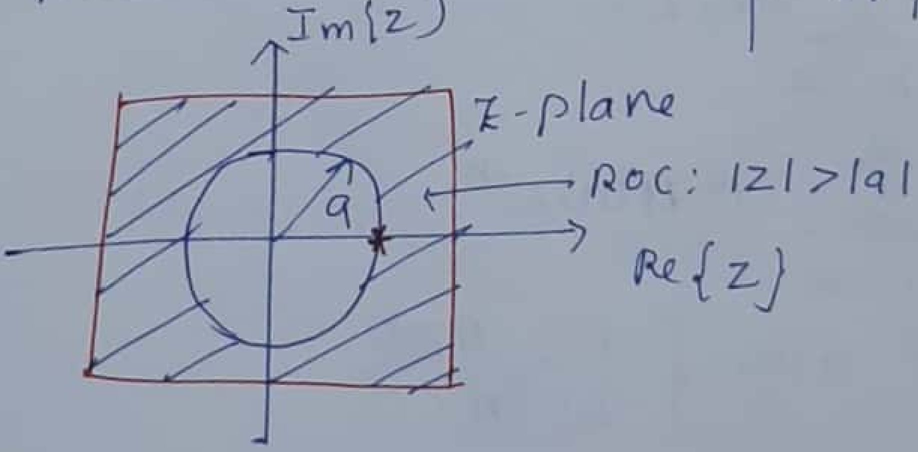
$$|a| < |z|$$

or $\boxed{|z| > |a|}$

$|\geq 0$

<0

$>$



zero is indicated by ⊙

pole → ——— $*$

one pole is at $z = a$

② Negative time exponential

$$x(n) = -a^n u(-n-1)$$

$$X(z) \triangleq \sum_{n=-\infty}^{-\infty} x(n) z^{-n}$$

$$= -\sum_{n=-\infty}^{\infty} a^n u(-n-1) z^{-n}$$

Wkt $u(-n-1) = \begin{cases} 0 ; n > -1 \\ 1 ; n \le -1 \end{cases}$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=-\infty}^{\infty} (az^{-1})^n$$

$$X(z) = -\sum_{n=-\infty}^{-1} \left(\frac{a}{z}\right)^{+n}$$

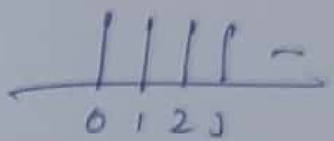
Let $n = -m$ ∴ above eqn becomes

$$X(z) = -\sum_{m=\infty}^{1} \left(\frac{z}{a}\right)^m \quad \left[\left(\frac{a}{z}\right)^{-m} = \left(\frac{z}{a}\right)^n\right]$$

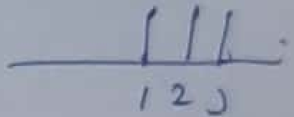
$$= -\sum_{m=1}^{\infty} \left(\frac{z}{a}\right)^m$$

$u(-n-1)$

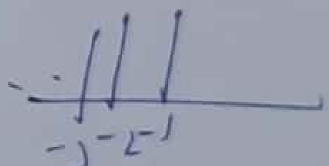
$u(n) = \begin{cases} 1, n \ge 0 \\ 0 ; n < 0 \end{cases}$



$u(n-1)$



$u(-n-1)$



$u(-n-1) = \begin{cases} 1, n \le -1 \\ 0 : n > 1 \end{cases}$

$$\boxed{\text{WKT} \sum_{n=1}^{\infty} a^n = \frac{a}{1-a} \; ; \; \text{if } |a|<1}$$

$a = \cancel{+}(z/a) \quad a = \left(\frac{z}{a}\right)$

~~X(z)~~ =

$$X(z) = \frac{-(z/a)}{1-(z/a)}$$

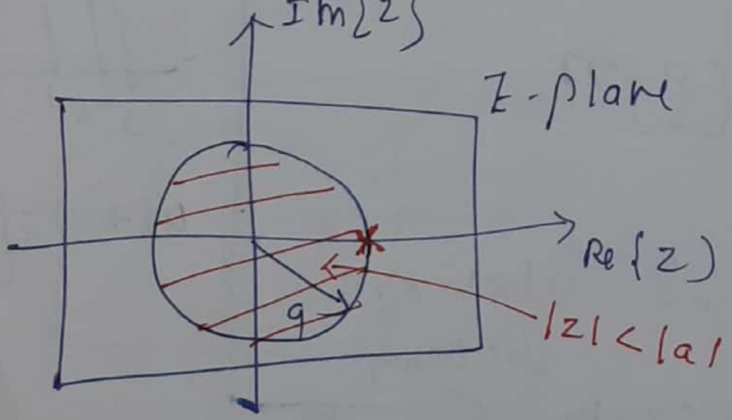
$$= \frac{-\left(\frac{z}{\cancel{a}}\right)}{\left(\frac{a-z}{\cancel{a}}\right)} = \frac{-z}{a-z}$$

ROC

$\left|\frac{z}{a}\right| < 1$

$|z| < |a|$

$$\boxed{X(z) = \frac{z}{z-a}} \quad \therefore \boxed{\text{ROC}: |z| < |a|}$$



③ double-sided ~~seq~~ exponential sequence

$x(n) = a^n u(n) - b^n u(-n-1)$

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [a^n u(n) - b^n u(-n-1)] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} -b^n u(-n-1) z^{-n} + \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$

$$= \frac{z}{z-b} + \frac{z}{z-a}$$

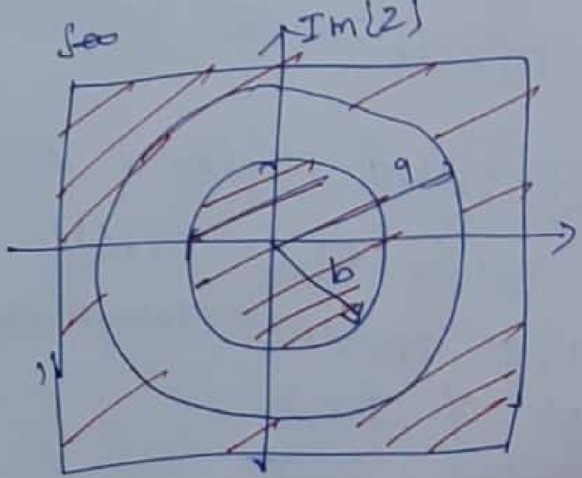
$(|z|<|b|) \cap (|z|>|a|)$

$$\boxed{X(z) = \frac{z}{z-a} + \frac{z}{z-b}}$$

ROC:

$(|z|>|a|) \cap (|z|<|b|)$

(If) $|a|>|b|$ eg $a=5$, $b=3$

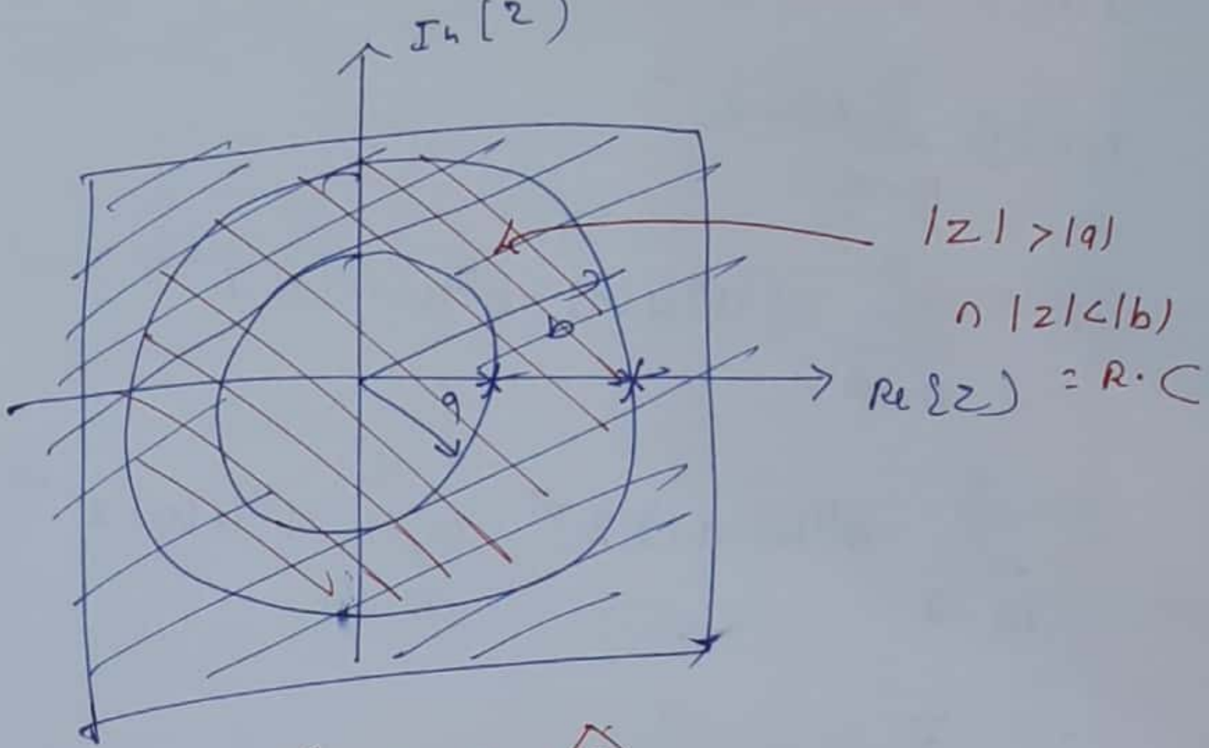


NO intersection

∴ ROC will not exist

(1) If $|a| < |b|$

$a = 3 \quad b = 5$



Properties of ROC

① ROC doesn't contain any poles

② If x(n) is finite duration seqn, then ROC is enitre Z-plance except at $Z = 0$ & $Z = \infty$

③ If x(n) is right sided infinite seqn, then ROC is ~~the~~ outside the outtermost circle. i.e. $|z| > \lambda_{max}$

④ If x(n) is leftsided infinite seqn, then ROC is inside the circle of radius equal to smallest magnitude of the pole of X(z)

$|z| < \lambda_{min}$

⑤ If $x(n)$ is double sided infinite ~~eqn~~, sequence then ROC will be the intersection of the ROC of the component

$$r_{min} < |z| < r_{max}$$

⑥ The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin

⑦ The ROC of LTI stable system contains unit circle in the z-plane.

Transforms of some useful sequence

① $x(n) = \delta(n)$

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n) z^{-n}$$

since $\delta(n)$ exists only at $n = 0$

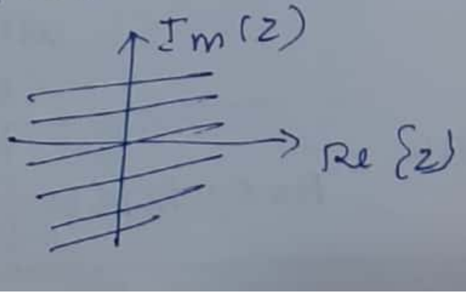
$$= \delta(0) z^{-0} = 1 \times z^{0} = 1 \times 1$$

$X(z) = 1$

$x(n) \xleftrightarrow{ZT} X(z)$

ROC = All z

$$\boxed{\delta(n) \longleftrightarrow 1 \; ; \; \text{Roc: } \forall z}$$



② $x(n) = \delta(n-n_0)$

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n-n_0) z^{-n}$$

$$= z^{-n}\Big|_{n=n_0} = \underline{\underline{z^{-n_0}}}$$

$$X(z) = z^{-n_0} = \frac{1}{z^{n_0}} \quad ; \quad |z| > 0$$

ROC = All z except at $z = 0$

$$\boxed{x(n) = \delta(n-n_0) \xleftrightarrow{ZT} z^{-n_0} \; ; \; \text{Roc} ; |z|>0}$$

③ $x(n) = \delta(n+n_0)$

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n+n_0) z^{-n}$$

$$= z^{-n}\Big|_{n=-n_0} = z^{n_0}$$

; Roc: $|z| < \infty$
all z value
except at $z = \infty$

$$\boxed{x(n) = \delta(n+n_0) \xleftrightarrow{ZT} z^{n_0} \; ; \; \text{Roc} ; |z| < \infty}$$

④ $x(n) = u(n)$

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} u(n) z^{-n}$$

WKT $u(n) = \begin{cases} 1 ; n \geq 0 \\ 0 ; n < 0 \end{cases}$

$$= \sum_{n=0}^{\infty} z^{-n}$$

$$= \sum_{n=0}^{\infty} (z^{-1})^{n}$$

WKT $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$

$|a| < 1$

$$= \frac{1}{1-z^{-1}} = \frac{z}{z-1} ;$$

ROC : $|z^{-1}| < 1$ $\left|\frac{1}{z}\right| < 1$

$\therefore$ $|z| > 1$

$$x(n) \xleftrightarrow{ZT} X(z)$$

$$\boxed{u(n) \xleftrightarrow{ZT} \frac{z}{z-1} ; \quad |z| > 1}$$

.X. ⑤ .X.

$x(n) = \sin \Omega_0 n \, u(n)$

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} \sin \Omega_0 n \, u(n) z^{-n}$$

⑧

$$X(z) = \sum_{n=0}^{\infty} \sin \Omega_0 n \; z^{-n}$$

$$= \sum_{n=0}^{\infty} \left[\frac{e^{j\Omega_0 n} - e^{-j\Omega_0 n}}{2j} \right] z^{-n}$$

$$= \frac{1}{2j} \left[\sum_{n=0}^{\infty} e^{j\Omega_0 n} . z^{-n} - \sum_{n=0}^{\infty} e^{-j\Omega_0 n} z^{-n} \right]$$

$$= \frac{1}{2j} \left[\sum_{n=0}^{\infty} \left[e^{j\Omega_0} z^{-1} \right]^n - \sum_{n=0}^{\infty} \left[e^{-j\Omega_0} z^{-1} \right]^n \right]$$

$$\boxed{\text{Wkt} \quad \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \; ; \quad |a| < 1}$$

$$= \frac{1}{2j} \left[\frac{1}{1 - e^{j\Omega_0} z^{-1}} - \frac{1}{1 - e^{-j\Omega_0} z^{-1}} \right]$$

$$= \frac{1}{2j} \left[\frac{z}{z - e^{j\Omega_0}} - \frac{z}{z - e^{-j\Omega_0}} \right]$$

$$= \frac{1}{2j}\left[\frac{z(z-e^{-j\Omega_0}) - z(z-e^{j\Omega_0})}{(z-e^{j\Omega_0})(z-e^{-j\Omega_0})}\right]$$

$$= \frac{1}{2j}\left[\frac{\cancel{z^2} - ze^{-j\Omega_0} - \cancel{z^2} + ze^{j\Omega_0}}{z^2 - ze^{-j\Omega_0} - ze^{j\Omega_0} + 1}\right]$$

$$= \frac{1}{2j}\left[\frac{\cancel{z^2}\, z\left[e^{j\Omega_0} - e^{-j\Omega_0}\right]}{z^2 - z\left[\underbrace{e^{-j\Omega_0} + e^{j\Omega_0}}_{=2\cos\Omega_0}\right] + 1}\right]$$

sin Ω0

$$\left[\begin{array}{l} \dfrac{e^{j\Omega_0} - e^{-j\Omega_0}}{2j} = \sin\Omega_0 \\ e^{-j\Omega_0} + e^{j\Omega_0} = 2\cos\Omega_0 \end{array}\right]$$

$$\boxed{X(z) = \frac{z\sin\Omega_0}{z^2 - 2z\cos\Omega_0 + 1}}$$

$$\boxed{\text{ROC: } |z| > 1}$$

ROC:

$|e^{j\Omega_0} z^{-1}| < 1$

& $|e^{-j\Omega_0} z^{-1}| < 1$

$|z^{-1}| < 1$ $\boxed{|z| > 1}$

⑥ $x(n) = \cos\Omega_0 n\, u(n)$

$$\boxed{X(z) = \frac{z^2 - z\cos\Omega_0}{z^2 - 2z\cos\Omega_0 + 1} \; ; \; \text{ROC: } |z| > 1}$$

Derive it

⑨

(ii) $x(n) = \cos\Omega_0 n \; u(n)$

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \cos\Omega_0 n \; z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\left[e^{j\Omega_0 n} + e^{-j\Omega_0 n}\right] z^{-n}$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty}\left(e^{j\Omega_0} z^{-1}\right)^n + \sum_{n=0}^{\infty}\left(e^{-j\Omega_0} z^{-1}\right)^n\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-e^{j\Omega_0}z^{-1}} + \frac{1}{1-e^{-j\Omega_0}z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{j\Omega_0}} + \frac{z}{z-e^{-j\Omega_0}}\right]$$

$$= \frac{1}{2}\left[\frac{z^2 - ze^{-j\Omega_0} + z^2 - ze^{j\Omega_0}}{z^2 - ze^{-j\Omega_0} - ze^{j\Omega_0} + 1}\right]$$

$$(z) = \frac{z^2 - z\cos\Omega_0}{z^2 - 2z\cos\Omega_0 + 1} \quad ; \quad \text{ROC}: |z| > 1$$

⑦ $x(n) = n u(n)$

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} n \cdot u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} n z^{-n} = \sum_{n=0}^{\infty} n (z^{-1})^{n} \Rightarrow$$

$$\boxed{\text{WKT} \sum_{n=0}^{\infty} n a^{n} = \frac{a}{(1-a)^2} \quad |a| < 1}$$

$a = z^{-1}$

$$X(z) = \frac{z^{-1}}{(1-z^{-1})^2} = \frac{z}{(z-1)^2} \quad |z| > 1$$

$$x(n) \xleftrightarrow{ZT} X(z)$$

$$\boxed{n u(n) \xleftrightarrow{ZT} \frac{z}{(z-1)^2}; \quad |z| > 1}$$

Properties of ZT

① Linearity

① If $x_1(n) \xleftrightarrow{ZT} X_1(z)$ with $ROC = R_1$
$= Rx_1^- < |z| < Rx_1^+$

& $x_2(n) \xleftrightarrow{ZT} X_2(z)$ with $ROC = R_2$
$: Rx_2^- < |z| < Rx_2^+$

then

$$x(n) = a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{ZT} a_1 X_1(z) + a_2 X_2(z) = X(z)$$

$ROC : R_1 \cap R_2$

Proof!

$$Z\{x(n)\} = X(z) \triangleq \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$Z\{a_1 x_1(n) + a_2 x_2(n)\}$

$$X(z) = \sum_{n=-\infty}^{\infty} [a_1 x_1(n) + a_2 x_2(n)] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a_1 x_1(n) z^{-n} + \sum_{n=-\infty}^{\infty} a_2 x_2(n) z^{-n}$$

$$= a_1 \underbrace{\sum_{n=-\infty}^{\infty} x_1(n) z^{-n}} + a_2 \underbrace{\sum_{n=-\infty}^{\infty} x_2(n) z^{-n}}$$

$$= a_1 X_1(z) + a_2 X_2(z)$$

② Time shifting

If $x(n) \xleftrightarrow{ZT} X(z)$ with $ROC: R$

then $x(n-n_0) \xleftrightarrow{ZT} z^{-n_0} X(z)$ with
$ROC = R$

Proof:

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$x(n) = x(n-no)$$

$$Z\{x(n-no)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n-no) z^{-n}$$

Put $n - no = m$ $\therefore n = m + no$

$$X(z) = \sum_{m=-\infty}^{\infty} x(m) z^{-(m+no)}$$

$$= \sum_{m=-\infty}^{\infty} x(m) z^{-m-no}$$

$$= \sum_{m=-\infty}^{\infty} x(m) z^{-m} . z^{\cdot no}$$

$$= z^{-no} \underbrace{\sum_{m=-\infty}^{\infty} x(m) z^{-m}}$$

$$Z\{x(n-no)\} = z^{-no} X(z)$$

③ multiplication by exponential or scaling in z-domain

If $x(n) \xleftrightarrow{ZT} X(z)$ with ROC = R

then

$$z\{a^n x(n)\} = X\left(\frac{z}{a}\right) \quad \text{ROC} = |a|R.$$

Proof $X(z) \triangleq \sum_{n=-\infty}^{\infty} x(n) z^{-n} = z\{x(n)\}$ — ①

$$z\{a^n x(n)\} = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{a^{-n}} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left(\frac{z}{a}\right)^{-n} \quad \text{— ②}$$

Comparing eq ① with ②

$$z\{a^n x(n)\} = X\left(\frac{z}{a}\right)$$

④ Time reversal

If $x(n) \xleftrightarrow{ZT} X(z)$ with ROC : R

then $x(-n) \xleftrightarrow{ZT} X\left(\frac{1}{z}\right)$ ROC : $\frac{1}{R}$

Proof :-

$$X(z) = Z\{x(n)\} \triangleq \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

Let $l = -n$ only on RHS

$$= \sum_{l=-\infty}^{\infty} x(l) \; z^{l}$$

$$= \sum_{l=-\infty}^{\infty} x(l) \left(z^{-1}\right)^{l}$$

$$= \sum_{l=-\infty}^{\infty} x(l) \left(\frac{1}{z}\right)^{l}$$

$$Z\{x(-n)\} = X\left(\frac{1}{z}\right)$$

Time expansion

⑤ multiplication by a ramp or differentiation in Z-domain

If $Z\{x(n)\} = X(z)$

$$x(n) \xleftrightarrow{ZT} X(z) \text{ with ROC} = R$$

then $$nx(n) \xleftrightarrow{ZT} -z\frac{dX(z)}{dz} \text{ with ROC} = R$$

Proof

$$X(z) = Z\{x(n)\} \triangleq \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \quad ①$$

differentiating WRT z on both sides

$$\frac{dX(z)}{dz} = \frac{d}{dz}\left\{\sum_{n=-\infty}^{\infty} x(n)\, z^{-n}\right\} = \sum_{n=-\infty}^{\infty} x(n)\, \frac{d}{dz} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, (-n) \underbrace{z^{-n-1}}_{z^{-n}.z^{-1}}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, (-n)\, z^{-1} z^{-n}$$

$$= \cancel{-nz^{-1}\sum_{n=}}$$

$$= -z^{-1} \sum_{n=-\infty}^{\infty} x(n) \cdot n \cdot z^{-n}$$

x'lying the eq' out by $-z$

$$-z\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} [n\,x(n)]\,z^{-n} \quad \text{②}$$

$$-z\frac{dX(z)}{dz} = Z\{n\,x(n)\} \quad \left[\text{comparing eq ② wrt ①}\right]$$

⑥ Convolution Property [Time domain]

If $Z\{x_1(n)\} = X_1(z)$ with $ROC = R_1$

& $Z\{x_2(n)\} = X_2(z)$ with $ROC = R_2$

then

$$Z\{x_1(n) * x_2(n)\} = X_1(z)\cdot X_2(z) \quad ROC = R_1 \cap R_2$$

Proof,

$$Z\{x(n)\} = X(z) \triangleq \sum_{n=-\infty}^{\infty} x(n)\,z^{-n}$$

$$Z\{x_1(n) * x_2(n)\} = \sum_{n=-\infty}^{\infty} [x_1(n) * x_2(n)]\,z^{-n} \quad \text{①}$$

WKT

$$x_1(n) * x_2(n) \triangleq \sum_{k=-\infty}^{\infty} x_1(k)\cdot x_2(n-k) \quad \text{②}$$

using eq ② in eq ①

$$= \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x_1(k)\cdot x_2(n-k)\right] z^{-n}$$

interchanging the order of summations

$$Z\{x_1(n) * x_2(n)\} = \sum_{K=-\infty}^{\infty} x_1(K) \left[\sum_{n=-\infty}^{\infty} x_2(n-K) \cdot z^{-n} \right]$$

Put $n-K = m$

$\therefore \; n = m+K$

$$= \sum_{K=-\infty}^{\infty} x_1(K) \cdot \sum_{m=-\infty}^{\infty} x_2(m) \; \underbrace{z^{-(m+K)}}_{z^{-m} \cdot z^{-k}}$$

$$= \sum_{K=-\infty}^{\infty} x_1(K) \cdot \sum_{m=-\infty}^{\infty} x_2(m) \, z^{-m} \cdot z^{-k}$$

$$= \underbrace{\sum_{K=-\infty}^{\infty} x_1(K) \, z^{-K}}_{} \; \underbrace{\sum_{m=-\infty}^{\infty} x_2(m) \, z^{-m}}_{}$$

$$Z\{x_1(n) * x_2(n)\} = X_1(Z) \cdot X_2(Z)$$

~~⑦ Time revers~~al

⑦ Accumulation

If $x(n) \xleftrightarrow{ZT} X(z)$ with ROC: R

then $\sum_{K=-\infty}^{n} x(K) \xleftrightarrow{ZT} \dfrac{X(z)}{1-z^{-1}}$ with ROC at least $R = R \cap \{|z|>1\}$

Proof:-

Let us consider convolution of a signal $x(n)$ with unit step function $u(n)$

$$x(n) * u(n) = \sum_{K=-\infty}^{\infty} x(K) . u(n-K) \longrightarrow ①$$

WKT

$$u(n-K) = \begin{cases} 1 ; & n-K \geq 0 \text{ (or) } \boxed{K \leq n} \\ 0 ; & K > n \end{cases}$$

$\therefore$ eq ① becomes

$$x(n) * u(n) = \sum_{K=-\infty}^{n} x(K) . \text{———} ②$$

Taking Z-Transforms on eq ②

$$Z\left\{x(n) * u(n)\right\} = Z\left\{\sum_{K=-\infty}^{n} x(K)\right\}$$

using convolution property &

ZT of $u(n) \longleftrightarrow \frac{Z}{Z-a}$ (or) $\frac{1}{1-Z^{-1}}$

in above eqn we get

$$X(Z) . \frac{1}{1-Z^{-1}} = Z\left\{\sum_{K=\infty}^{n} x(K)\right\}$$

Rewritting,

$$Z\left\{\sum_{K=-\infty}^{n} x(k)\right\} = \frac{X(Z)}{1-Z-1}$$

Z-Transforms of some usefull seqn

		Z-transform	RoC
① unit step.	$u(n)$	$\frac{z}{z-1}$	$\lvert z\rvert > 1$
	$-u(-n-1)$	$\frac{z}{z-1}$	$\lvert z\rvert < 1$
② unit impulse	$\delta(n)$	1	All z
	$\delta(n-n_0)$	z^{-n_0}	$\lvert z\rvert > 0$
	$\delta(n+n_0)$	z^{n_0}	$\lvert z\rvert < \infty$
③ unit ramp	$n\,u(n)$	$\frac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
④ sinusoidal	$\sin\Omega_0 n\, u(n)$	$\frac{z\sin\Omega_0}{z^2-2z\cos\Omega_0+1}$	$\lvert z\rvert > 1$
	$\cos\Omega_0 n\, u(n)$	$\frac{z^2-z\cos\Omega_0}{z^2-2z\cos\Omega_0+1}$	$\lvert z\rvert > 1$
Exponential	$a^n u(n)$	$\frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
	$-b^n u(-n-1)$	$\frac{z}{z-b}$	$\lvert z\rvert < \lvert b\rvert$

Properties

Property		Z-transf	Roc
(1) Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(z) + a_2 X_2(z)$	$R_1 \cap R_2$
(2) shifting property (or) Time shift or Translation	$x(n-n_0)$	$z^{-n_0} X(z)$	R_1
(3) multiplication by an exponential (or) scaling property [scaling in z-domain]	$a^n x(n)$	$X\left(\frac{z}{a}\right)$	$\lvert a \rvert R$
(4) multiplication by a ramp (or) differentiation in z-domain	$n\, x(n)$	$-z \frac{dX(z)}{dz}$	R
(5) convolution	$x(n) * y(n)$	$X(z) Y(z)$	$R_1 \cap R_2$
(6) Time reversal	$x(-n)$	$X\left(\frac{1}{z}\right)$	$z \in \frac{1}{R_x}$

Problems

Find the z-transforms of the foll signals

i) $x(n) = \alpha^{-n} u(-n-1)$

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \alpha^{-n} u(-n-1) z^{-n}$$

$$= \sum_{n=-\infty}^{-1} (\alpha z)^{-n})$$

Let $n = -m$

$$= \sum_{m=+\infty}^{1} (\alpha z)^{m} = \sum_{m=1}^{\infty} (\alpha z)^{m} + 1 - 1$$

$$= \sum_{m=0}^{\infty} (\alpha z)^{m} - 1$$

$|\alpha z| < 1$

$|z| < \frac{1}{|\alpha|}$

$$= \frac{1}{1-\alpha z} - 1$$

$$= \frac{1 - 1 + \alpha z}{1-\alpha z} = \frac{+\alpha z}{1-\alpha z} = \frac{+\alpha z}{\alpha[1/\alpha - z]}$$

$$= \frac{-z}{(z - 1/\alpha)}$$

⑧ $x(n) = a^n \cos(\Omega_0 n) \cdot u(n)$

WKT

$$Z\{\cos(\Omega_0 n)\, u(n)\} = \frac{z^2 - z\cos\Omega_0}{z^2 - 2z\cos\Omega_0 + 1}$$

; ROC $|z| > 1$

by multiplication property.

$$Z\{a^n x(n)\} = X\left(\frac{z}{a}\right) \text{ or } X(a^{-1}z)$$

hence

$Z\{a^n \cos(\Omega_0 n) \cdot u(n)\}$

replace z by z/a

$$= \frac{\left(\frac{z}{a}\right)^2 - \left(\frac{z}{a}\right)\cos\Omega_0}{\left(\frac{z}{a}\right)^2 - 2\left(\frac{z}{a}\right)\cos\Omega_0 + 1}$$

$$= \frac{\left(\frac{z}{a}\right)^2 [1 - az^{-1}\cos\Omega_0]}{\left(\frac{z}{a}\right)^2 [1 - 2az^{-1}\cos\Omega_0 + a^2z^{-2}]}$$

$$= \frac{1 - az^{-1}\cos\Omega_0}{1 - 2az^{-1}\cos\Omega_0 + a^2z^{-2}}$$

ROC: $\left|\frac{z}{a}\right| > 1$

$|z| > |a|$

3) ~~$x(n) =$~~

3) $x(n) = \left[4\left(\frac{1}{2}\right)^n + 5(2)^n\right]u(n)$

$= 4\left(\frac{1}{2}\right)^n u(n) + 5\,(2)^n u(n)$

$x(n) = 4\,x_1(n) + 5\,x_2(n)$.

. to find $X_1(z)$

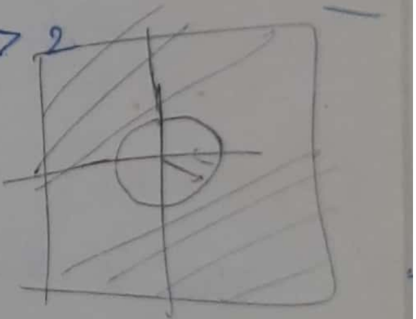
Let $x_1(n) = \left(\frac{1}{2}\right)^n u(n)$

WKT $Z\left[a^n u(n)\right] = \frac{z}{z-a}$ $|z| > |a|$

$X_1(z) = \frac{z}{z-(1/2)}$ ROC: $|z| > \frac{1}{2}$

Let $x_2(n) = (2)^n u(n)$

$X_2(z) = \frac{z}{z-2}$ ROC: $|z| > 2$

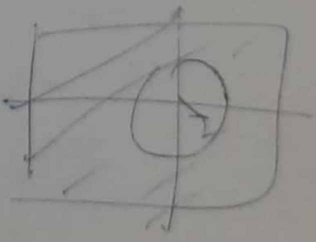


$x(n) = 4\,x_1(n) + 5\,x_2(n)$

taking Z-transform

$X(z) = 4\,X_1(z) + 5\,X_2(z)$

$= \frac{4z}{z-1/2} + \frac{5z}{z-2}$



ROC: $(|z| > 1/2) \cap (|z| > 2)$

$= |z| > 2$

② $x(n) = 7(2)^n u(n) - 3(5)^n u(-n-1)$

$a^n u(n) \leftrightarrow \frac{Z}{Z-a}$

$|z| > |a|$

Let $x_1(n) = (2)^n u(n)$

$X_1(z) = \frac{z}{z-2}$; $|z| > 2$

$x_2(n) = -(5)^n u(-n-1)$

$-b^n u(-n-1) = \frac{Z}{z-b}$

$|z| < |b$

$= \frac{Z}{z-5}$ $|z| < 5$

Let $x(n) = 7x_1(n) + 3x_2(n)$

$= \frac{7z}{z-2} + \frac{3z}{z-5}$

Roc: $(|z| > 2) \cap (|z| < 5)$

$2 < |z| < 5$

③ $x(n) = e^{j\Omega_0 n} u(n) = a^n u(n)$

This is of the form $a^n u(n)$ where $\underline{a = e^{j\Omega_0}}$ $\Longrightarrow$

wkt $Z\{a^n u(n)\} = \frac{z}{z-a}$ $|z| > |a|$

$\therefore X(z) = \frac{Z}{Z - e^{j\Omega_0}}$ $|z| > 1$

④ $x(n) = \{2, 3, \underset{\uparrow}{5}, 7, 8, 9\}$

$$X(z) = \sum_{n=-2}^{3} x(n)\, z^{-n}$$

$$= 2z^{2} + 3z + 5 + 7z^{-1} + 8z^{-2} + 9z^{-3}$$

ROC: All z except at 0 & ∞

⑤ $x(n) = \left(\frac{1}{2}\right)^{n} \sin \Omega_0 n \; u(n)$

WKT $\sin \Omega_0 n \, u(n) \longleftrightarrow \frac{z \sin \Omega_0}{z^{2} - 2z\cos\Omega_0 + 1}$ ROC $|z|>1$

& scaling property

$$a^{n} x(n) = X\left(\frac{z}{a}\right) \quad \text{ROC}: |a| R$$

z is replaced by $z/\frac{1}{2}$. $2z$

$$\left(\frac{1}{2}\right)^{n} \sin \Omega_0 n \; u(n) = \frac{2z \sin \Omega_0}{4z^{2} - 4z\cos\Omega_0 + 1}$$

$$\text{ROC}: \left|\frac{z}{1/2}\right| > 1$$

$$|z| > 1/2$$

⑥ $x(n) = n\,u(n)$

WKT $u(n) \xleftrightarrow{ZT} \frac{z}{z-1}$ ROC: $|z|>1$

Apply differentiation in z-domain property

$$n\,x(n) = -z \frac{dX(z)}{dz}$$

$$Z\{n\,u(n)\} = -z\frac{d}{dz}\left[\frac{z}{z-1}\right]$$

$$= -z\left[\frac{(z-1)\frac{d}{dz}z - z\frac{d}{dz}(z-1)}{(z-1)^2}\right]$$

$$= -z\left[\frac{(z-1)1 - z(1)}{(z-1)^2}\right]$$

$$= \frac{-z\,[\cancel{z}-1-\cancel{z}]}{(z-1)^2}$$

$$= \frac{z}{(z-1)^2}$$ ROC: $|z|>1$

⑦ $x(n) = n^2 . u(n)$

$$u(n) = \frac{z}{(z-1)}$$

$$nu(n) = -z\frac{d}{dz}\left(\frac{z}{z-1}\right) = \frac{z}{(z-1)^2}$$

$$n^2u(n) = -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$= \frac{z\,[z+1]}{(z-1)^3}$$ ROC: $|z|>1$

⑧ $x(n) = n a^n u(n)$

wkt $u(n) \longleftrightarrow \frac{z}{z-1}$ ROC: $|z| > 1$

using scaling proper

$a^n u(n) \Leftrightarrow \frac{z}{z-a}$ $|z| > |a|$

using differentiation property

$n a^n u(n) = -z \frac{d}{dz}\left[\frac{z}{z-a}\right]$

$= \frac{a z}{(z-a)^2}$ ROC: $|z| > |a|$

⑨ $x(n) = u(-n)$

wkt $u(n) = \frac{z}{z-1}$ $|z| > 1$

using time reversal proper

$x(-n) \longleftrightarrow X\left(\frac{1}{z}\right)$ ROC: $z \in \frac{1}{R_x}$

$\therefore u(-n) \longleftrightarrow \frac{1/z}{1/z - 1}$ $|z| < 1$

$= \frac{z}{z(1-z)} = \frac{1}{(1-z)}$

⑧ $x(n) = \left[n\left(-\frac{1}{2}\right)^n u(n)\right] * \left[\left(\frac{1}{4}\right)^{-n} u(-n)\right]$

$x_1(n) \quad * \quad x_2(n)$

Soln.

(1) first let us find Z-transforms of

$x_1(n) = n\left(-\frac{1}{2}\right)^n u(n)$

WKT $a^n u(n) \Longleftrightarrow \frac{z}{z-a}$ R.C : $|z| > |a|$

$\therefore \left(-\frac{1}{2}\right)^n u(n) = \frac{z}{z+1/2}$ ROC : $|z| > \frac{1}{2}$

using Z-domain differentiation property

$n.x(n) \longleftrightarrow -z \frac{d\,x(n)}{dz}$

$x_1(n) = n.(-1/2)^n u(n) \xleftrightarrow{z} -z \frac{d}{dz}\left(\frac{z}{z+1/2}\right)$

$= \frac{-1/2\, z}{(z+1/2)^2}$ with ROC: $|z| > 1/2$

(2) next find Z-transform of

$x_2(n) = \left(\frac{1}{4}\right)^{-n} u(-n)$

WKT $\left(\frac{1}{4}\right)^n u(n) = \frac{z}{z-1/4}$ with ROC : $|z| > \frac{1}{4}$

use time reversal property

(17)

$x(-n) \xleftrightarrow{z} X(z^{-1})$ with RoC $l \frac{1}{Rx}$

$X_2(z) : \left(\frac{1}{4}\right)^{-n} u(-n) \xleftrightarrow{z} = \frac{z^{-1}}{z^{-1}-1/4}$

$= \frac{-4}{(z-4)}$

with ROC : $\left|\frac{1}{z}\right| > \frac{1}{4}$

$|z| < 4$

we apply convolution property to obtain

$X(z) = X_1(z) \cdot X_2(z)$

$= \frac{-\frac{1}{2}z}{(z+1/2)^2} \cdot \frac{-4}{(z-4)}$

$= \frac{2z}{(z-4)(z+1/2)^2}$

(9) $x(n) = a^{n-1} u(n-1)$

wkt $a^n u(n) \xleftrightarrow{z} \frac{z}{z-a}$ RoC $|z| > |a|$

using time shifting property

$Z\{x(n-n_0)\} = z^{-n_0} X(z)$

$$Z\{a^{n-1}u(n-1)\} = z^{-1}\frac{z}{z-a}$$

here $n_0 = 1$ $$= \frac{1}{z-a}$$ ROC: $|z|>|a|$

② determine the signal $x(n)$ whose z-transform is given by
$X(z) = \log(1-az^{-1})$; $|z|>|a|$

$$X(z) = \log(1-az^{-1})$$

$$\frac{d}{dz}X(z) = \frac{1}{1-az^{-1}}(az^{-2}) = \frac{az^{-2}}{1-az^{-1}}$$

x'ly both the sides by $-z$

$$-z\frac{d}{dz}X(z) = \frac{-az^{-1}}{1-az^{-1}}$$

$$= -az^{-1}\left[\frac{1}{1-az^{-1}}\right]$$

$$= -az^{-1}\left[\frac{z}{z-a}\right]$$

$$= -az^{-1}\left[\cancel{a}Z[a^nu(n)]\right]$$

$$= -a\,Z\left[a^{n-1}u(n-1)\right] \text{ --- } ①$$

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from differentiation property

$$\overset{\cup l a}{Z\{n x(n)\}} : -z \frac{d}{dz} X(z) \text{ --- } ②$$

comparing eq ① & ②

$$n\, x(n) : -a\,[a^{n-1} u(n-1)]$$

$$x(n) : -\frac{a\,[a^{n-1} u(n-1)]}{n}$$

$$: -\frac{a^n u(n-1)}{n}$$

find Z-transform

① ~~unit step to~~ unit step

(i) $x(n) = n \sin\left(\frac{\pi n}{2}\right) u(-n)$

$= -n \sin\left(-\frac{\pi n}{2}\right) u(-n)$

$= x_1(-n)$

$x_1(n) = n \sin\left(\frac{\pi n}{2}\right) u(n)$

$= n\, x_2(n)$

$x_2(n) = \sin\left(\frac{\pi n}{2}\right) u(n)$

$$X_2(z) = \frac{z \sin \pi/2}{z^2 + 2z\cos\frac{\pi}{2} + 1}, \quad |z| > 1$$

$$= \frac{z}{z^2+1}, \quad |z| > 1$$

$$X_1(z) = -z \frac{d}{dz}(X_2(z))$$

$$= -z \frac{d}{dz}\left(\frac{z}{z^2+1}\right)$$

$$= -z\left[\frac{(z^2+1)(1) - z(2z)}{(z^2+1)^2}\right]$$

$$= \cancel{z^{3}2}$$

$$= -z\left[\frac{z^2+1-2z^2}{(z^2+1)^2}\right]$$

$$= \frac{z^3 - z}{(z^2+1)^2} \quad |z|>1$$

$$X(z) = X_1(1/z)$$

$$= \frac{\left(\frac{1}{z}\right)^3 - \left(\frac{1}{z}\right)}{\left(\left(\frac{1}{z}\right)^2 + 1\right)^2}$$

$$= \frac{\frac{1}{z^3}\left[1 - z^2\right]}{\frac{1}{z^4}\left[1 + z^2\right]^2}$$

$$= \frac{z\left(1 - z^2\right)}{\left(1 + z^2\right)^2} \qquad |z| < 1$$

(b) $x(n) = [n-2]\left(\frac{1}{2}\right)^n u(n-2)$

$= (n-2)\left(\frac{1}{2}\right)^{n-2+2} u(n-2)$

$= (n-2)\left(\frac{1}{2}\right)^{n-2} \cdot \left(\frac{1}{2}\right)^2 u(n-2)$

$= (n-2)\left(\frac{1}{2}\right)^{n-2} u(n-2) \times \frac{1}{4}$

$x(n) = \frac{1}{4} x_1(n-2)$

$x_1(n) = n\left(\frac{1}{2}\right)^n u(n)$

$= \left(\frac{1}{2}\right)^n x_2(n)$

$x_2(n) = n\,u(n)$

$X_2(z) = \frac{z}{(z-1)^2}$; $|z| > 1$

$X_1(z) = X_2\left(\frac{z}{a}\right)$

$$= X_2\left(\frac{z}{1/2}\right)$$

$$= X_2(2z)$$

$$\therefore X_1(z) = \frac{2z}{(2z-1)^2} \qquad |2z| > 1$$

$$|z| > 1/2$$

$$X\cancel{(z)} =$$

$$x(n) = \frac{1}{4} x_1(n-2)$$

$$X(z) = z^{-2} \times \frac{1}{4} X_1(z)$$

$$= \frac{1}{4} z^{-2} \times \frac{2z}{(2z-1)^2}$$

$$= \frac{1}{\underset{2}{\cancel{4}}} \times \frac{\cancel{2}z}{z^2(2z-1)^2}$$

$$= \frac{1}{2z(2z-1)^2} \; ; \quad |z| > 1/2$$

3) $n(n-1)\, a^n u(n)$

$[n^2 - n]\, a^n u(n)$

$n^2 a^n u(n) - n a^n u(n)$

$\because \; a^n u(n) \longleftrightarrow \frac{z}{z-a}$

$n \cdot a^n u(n) \longleftrightarrow \frac{za}{(z-a)^2}$

$n^2 \cdot a^n u(n) \longleftrightarrow \frac{2za^2}{(z-a)^3}$

4) $a^n \cos \Omega_0 n \; u(n)$

$$= \frac{\left(\frac{z}{a}\right)^2 - \left(\frac{z}{a}\right) \cos \Omega_0}{(z/a)^2 - 2(z/a)\cos \Omega_0 + 1}$$

① $h(n) = (1/3)^n u(n) + \left(\frac{1}{2}\right)^{n-2} u(n-1)$

$= \left(\frac{1}{3}\right)^n u(n) + \left(\frac{1}{2}\right)^{n-1} \cdot 2\, u(n-1)$

$(1/2)^n u(-n-1) \quad - \left(-\frac{1}{3}\right)^n u(-n-1)$

3) Find the conv of $x_1(n) = \{2, \underset{\uparrow}{3}, 4\}$ &

$x_2(n) = \{\underset{\uparrow}{1}, 5, 9\}$ usi

Z tra

$X_1(z) = 2z + 3 + 4z^{-1}$

$X_2(z) = 1 + 5z^{-1} + 9z^{-2}$

$Y(z) = X_1(z) \cdot X_2(z)$

$= (2z + 3 + 4z^{-1})(1 + 5z^{-1} + 9z^{-2})$

$= 2z + 13 + 37z^{-1} + 47z^{-2} + 36z^{-3}$

IZT

$= \{2, \underset{\uparrow}{13}, 37, 47, 36\}$

Initial value theorem:-

If $x(n)$ is causal &

$Z\{x(n)\}$ is $X(z)$

then $x[0] = \underset{z\to\infty}{Lt} X(z)$.

Proof:- by defn of ZT

$$Z\{x(n)\} = X(z) \triangleq \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

since $x(n)$ is causal

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= x(0) + x(1) z^{-1} + x(2) z^{-2} + \dots$$

Taking $\underset{z\to\infty}{Lt}$ on both sides

$$\underset{z\to\infty}{Lt} [X(z)] = \underset{z\to\infty}{Lt} \Big[x(0) + x(1) \underset{z\to\infty}{\cancel{Lt}} z^{-1} + x(2) z^{-2} \dots \Big]$$

$$\boxed{\underset{z\to\infty}{Lt} X(z) = x(0)}$$

hence proof

Final value theorem:-

If $x(n)$ is causal & $x(n) \longleftrightarrow X(z)$

then $\underset{N\to\infty}{Lt} x(N) = x(\infty) = \underset{z\to 1}{Lt} [1 - z^{-1}] X(z)$

proof:

$$Z\{x(n) - x(n-1)\} = \sum_{n=0}^{\infty} [x(n) - x(n-1)] z^{-n}$$

$$= X(z) - X(z)\, z^{-1}$$

$$\sum_{n=0}^{\infty} [x(n) - x(n-1)] z^{-n} = X(z)\,[1 - z^{-1}]$$

Taking limits $n \to \infty$ & $z \to 1$ on both sides

$$\underset{z\to 1}{Lt}\ \underset{n\to\infty}{Lt} \sum_{n=0}^{\infty} [x(n) - x(n-1)] z^{-n} = \underset{z\to 1}{Lt}\ \underset{n\to\infty}{Lt}\ X(z)\,[1 - z^{-1}]$$

$$\underset{n\to\infty}{Lt} \sum_{n=0}^{\infty} x(n) - x(n-1) = \underset{z\to 1}{Lt}\ X(z)\,[1 - z^{-1}] \rightarrow ①$$

Note: $$\sum_{n=0}^{5} x(n) - x(n-1) = \sum_{n=0}^{5} x(n) - \sum_{n=0}^{5} x(n-1)$$

$$= x(0) + x(1) + x(2) + x(3) + x(4) + x(5)$$
$$- x(-1) - x(0) - x(2) - x(3) - x(4$$
$$- x(1)$$

$$= x(5)$$

$$\sum_{n=0}^{N} x(n) - x(n-1) = x(N)$$

$$\underset{n\to\infty}{Lt} \sum_{n=0}^{\infty}[x(n)-x(n-1)] = \underset{z\to 1}{Lt}\, X(z)[1-z^{-1}]$$

$$\boxed{\underset{N\to\infty}{Lt}\, x(N) = x(\infty) = \underset{z\to 1}{Lt}\, [1-z^{-1}]X(z)}$$

hence proved

Inverse Z - transforms :

* The process of finding $x(n)$ from its Z-transform $X(z)$ is called the inverse Z-transform & is denoted as follows :

$$x(n) = z^{-1}[X(z)]$$

* some of the Methods used to find inverse Z-transform are :

(1) Long division method

(2) partial- fraction method

& (3) contour integral method

(1) Long -division or power series method :

* In this method expand given $X(z)$ into a power series of the form

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}.$$

* when $X(z)$ is rational, expansion is performed by long division.

* The division way depends on ROC

* If ROC is exterior (outside) to the circle, it represents right-sided seq n, then divide X(z) to get negative powers of z
[i.e. z^{-1}, z^{-2} . . .]

* If ROC is interior to the circle, it represents left sided seq n, then divide X(z) to get positive powers of z [i.e, z, z^{2} . . .]

① Find the inverse z-transforms of X(z) for the given ROC's using long division method

1) $X(z) = \frac{z}{z-a}$ (i) ROC $|z| > |a|$
(ii) ROC $|z| < |a|$

Soln;

(i) ROC $|z| > |a|$

$$\begin{array}{r|l} & 1 + az^{-1} + a^{2}z^{-2} + a^{3}z^{-3} + \cdots \\ \hline z-a & z \\ & z-a \\ \hline & a \\ & a - a^{2}z^{-1} \\ \hline & a^{2}z^{-1} \\ & a^{2}z^{-1} - a^{3}z^{-3} \\ \hline & a^{3}z^{-3} \end{array}$$

$\therefore X(z) = 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \dots$

$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

we can write the above eq^n as

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$

$\therefore$

$$x(n) = 1 + az^{-1} + a^2z^{-2} + \dots$$

$$= a^n \cdot u(n).$$

(ii) <u>ROC: $|z| < |a|$</u>

divide $X(z)$ to get the powers of z

express $X(z)$ in terms of increasing powers of z in both numerator & denominator

$$X(z) = \frac{z}{-a+z}$$

$$\begin{array}{r|l} & -a^{-1}z - a^{-2}z^2 - a^{-3}z^3. \\ \hline -a+z & z \\ & z - a^{-1}z^2 \\ \hline & a^{-1}z^2 \\ & a^{-1}z^2 - a^{-2}z^3 \\ \hline & a^{-2}z^3 \\ & \vdots \end{array}$$

$$X(z) = -a^{-1}z - a^{-2}z^{2} - a^{-3}z^{3} \cdot \cdot$$

$$= \sum_{n=1}^{\infty} -a^{-n}z^{n}$$

$$= -\sum_{n=-\infty}^{-1} a^{n}z^{-n}$$

$$= -\sum_{n=-\infty}^{\infty} a^{n}u(-n-1)\,z^{-n}$$

$$\therefore x(n) = -a^{n}u(-n-1)$$

② $X(z) = \dfrac{z}{z-3}$, ROC (i) $|z| > 3$ (ii) $|z| < 3$

(i) $|z| > 3$

$$\begin{array}{r|l} & 1+3z^{-1}+9z^{-2}+27z^{-3} \\ \hline z-3 & z \\ & z-3 \\ \hline & 3 \\ & 3-9z^{-1} \\ \hline & 9z^{-1} \\ & 9z^{-1}-27z^{2} \\ \hline & 27z^{-1} \end{array}$$

$$\therefore X(z) = 1+3z^{-1}+9z^{2}+27z^{-3}+$$

$$= \sum_{n=0}^{\infty} (3)^{n}z^{-n} = \sum_{n=-\infty}^{\infty} (3)^{n}u(n)\,z^{-n}$$

$$\therefore x(n) = (3)^{n}u(n) \;//$$

(ii) ROC $|z| < 3$

~~$-3^{-1}z \pm 9^{-2}z^2$~~

~~$-3+z \,\big|\, z$~~

~~$z - 3^{-1}z^2$~~

~~$3^{-1}z^2$~~

~~$3^{-1}z^2 \pm 9^{-1}z^3$~~

~~$9^{-1}z^3$~~

$$
\begin{array}{r|l}
 & -3^{-1}z - 3^{-2}z^2 - 3^{-3}z^3 + \cdots \\
\hline
-3+z & z \\
 & z - 3^{-1}z^2 \\
\hline
 & 3^{-1}z^2 \\
 & 3^{-1}z^2 - 3^{-2}z^3 \\
\hline
 & 3^{-2}z^3 \\
 & 3^{-2}z^3 - 3^{-3}z^4 \\
\hline
\end{array}
$$

$$X(z) = -\left[3^{-1}z + 3^{-2}z^2 + 3^{-3}z^3 + \cdots\right]$$

$$= -\sum_{n=1}^{\infty} 3^{-n}z^{n}$$

$$= -\sum_{n=-\infty}^{-1} 3^{n}z^{-n} \;:\; = -\sum_{n=-\infty}^{+\infty} 3^{n}u(-n-1)\,z^{-n}$$

$$\boxed{x(n) = -3^{n}u(-n-1)}$$

~~③ $X(z) = \frac{1+2z^{-1}}{1-2z^{-1}+z^{-2}}$~~

③ $X(z) = \frac{z}{z^2-2z+1}$

with ROC (i) $|z|>1$ & (ii) $|z|<1$

(i) $|z|>1$

$$\begin{array}{r|l} & z^{-1}+2z^{-2}+3z^{-3}+\cdots \\ \hline z^2-2z+1 & z \\ & z-2+z^{-1} \\ \hline & 2-z^{-1} \\ & 2-4z^{-1}+2z^{-2} \\ \hline & 3z^{-1}-2z^{-2} \end{array}$$

$X(z) = z^{-1}+2z^{-2}+3z^{-3}+\cdots$

$= \sum_{n=0}^{\infty} nz^{-n}$

$= \sum_{n=-\infty}^{\infty} nu(n)z^{-n}$

$\therefore x(n) = nu(n)$

$x(n) = \sum_{k=0}^{\infty} \frac{(-4)^k}{2k!} \delta(n+2k)$

⑥ $X(z) = \cos(2z)$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2k!}$

$X(z) = \cos 2z$

$= \sum_{k=0}^{\infty} (-1)^k \frac{(2z)^{2k}}{(2k)!}$

$= \sum (-1)^k \frac{2^{2k} \cdot z^{2k}}{2k!}$

$= \sum (-4)^k \frac{z^{2k}}{2k!}$

$x(n) = \sum_{0}^{\infty} \frac{(-4)^k}{(2k)!} \delta(n+2k)$

(ii) ROC : $|z| < 1$

$$\begin{array}{r|l} & z + 2z^2 + 3z^3 + \cdots \\ \hline 1 - 2z + z^2 & z \\ & z - 2z^2 - z^3 \\ \hline & 2z^2 + z^3 \\ & \overset{(-)}{2z^2} \overset{(+)}{-4z^3} \overset{(+)}{+2z^4} \\ \hline & 3z^3 + 2z^4 \\ & \overset{(-)}{3z^3} \overset{(+)}{-6z^4} \overset{(-)}{+3z^5} \\ \hline & 4z^4 - 3z^5 \end{array}$$

$X(z) = z + 2z^2 + 3z^3 + \cdots$

$= \sum_{n=0}^{\infty} n z^n = \sum_{n=-\infty}^{0} (-n) z^{-n}$

$= \sum_{n=-\infty}^{\infty} (-n) u(-n) z^{-n}$

$= (-n) u(-n)$

④ $X(z) = e^{z^{-1}}$ ROC all z except $z = 0$

Soln :

WKT $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$

given $X(z) = e^{z^{-1}}$

here $x = z^{-1}$

$$\therefore X(z) = \sum_{n=0}^{\infty} \frac{(z^{-1})^n}{n!} = \sum_{n=-\infty}^{\infty} \frac{1}{n!} u(n) z^{-n}$$

$$\therefore x(n) = \frac{1}{n!} u(n)$$

② $X(z) = \log(1+az^{-1})$ ROC $|z| > |a|$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

$$X(z) = \log(1+az^{-1})$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} a^n z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1}}{n} a^n u(n-1) z^{-n}$$

$$\therefore x(n) = \frac{(-1)^{n+1}}{n} a^n u(n-1)$$

partial fraction expansion method

- Let $X(z)$ be a rational fun of z, then $x(n)$ can be found from $X(z)$ by using partial fraction expansion approach.

Let

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}}$$

if $M < N$ then the rational fun is called proper

- if $M \geq N$, then use long division to express $X(z)$ in the form

$$X(z) \sum_{K=0}^{M-N} f_K z^{-K} + \frac{\tilde{B}(z)}{A(z)}$$

The numerator polynomial $\tilde{B}(z)$ has order on less than that of denominator polynomial. Then partial fraction expansion method is applied to determine IZT of $\tilde{B}(z)/A(z)$

Some inverse Z-transforms useful for partial fraction meth

	$X(z)$	$x(n) = z^{-1}[X(z)]$ for $\|z\|>\|a\|$
1.	$\frac{z}{z-a}$	$a^n u(n)$
2.	$\frac{z}{(z-a)^2}$	$n a^{n-1} u(n)$
3.	$\frac{z}{(z-a)^3}$	$\frac{n(n-1) a^{n-2}}{2!} u(n)$

$$\frac{z}{(z-a)^K} = \frac{n(n-1) \dots (n-(K-2)) a^{n-K+1}}{(K-1)!} u(n)$$

Find the IZT of X(z) for the given ROCs

(1) $X(z) = \frac{z}{z^2-5z+6}$ ROC (i) $|z|>3$
(ii) $|z|<2$

Soln:

$$X(z) = \frac{z}{z^2-5z+6}$$

$s^2-5s+6=0$
$s^2-3s-2s+6=0$
$s(s-3)-2(s-3)=0$
$(s-3)(s-2)=0$

$$\frac{X(z)}{z} = \frac{1}{z^2-5z+6}$$

$$= \frac{1}{(z-3)(z-2)}$$

$$= \frac{A}{z-3} + \frac{B}{z-2}$$

~~$\frac{1}{(z-3)(z-2)} = \frac{A}{z-3} + \frac{B}{z-}$~~

$z =$ ~~$A(z-2)+B(z-$~~

$$A = \cancel{(z-3)} \cdot \frac{1}{\cancel{(z-3)}(z-2)}\Bigg|_{z=3}$$

$$= \frac{1}{z-2}\Bigg|_{z=3} = \frac{1}{3-2} = \underline{\underline{1}}$$

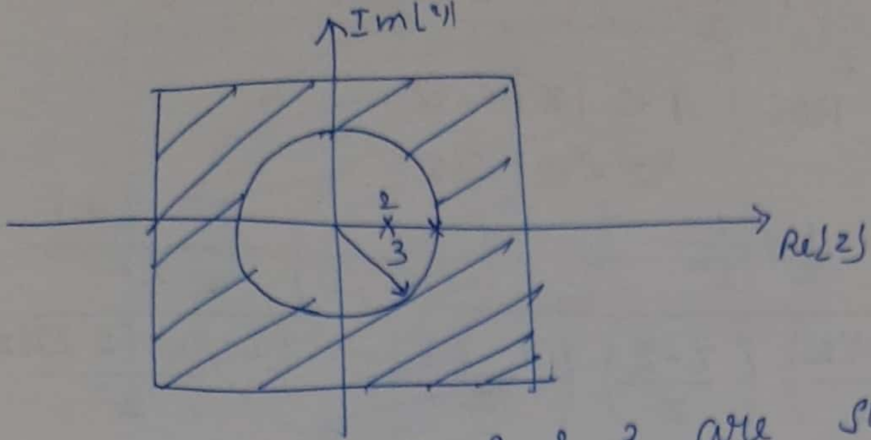
$$B = \frac{1}{(z-3)(z-2)}(z-2)\Bigg|_{z=2} = \frac{1}{2-3} = \underline{\underline{-1}}$$

$$\frac{X(z)}{z} = \frac{1}{z-3} - \frac{1}{z-2}$$

$$\therefore X(z) = \frac{z}{z-3} - \frac{z}{z-2}$$

(1) ROC : $|z| > 3$

poles: $z = 3$, $z = 2$



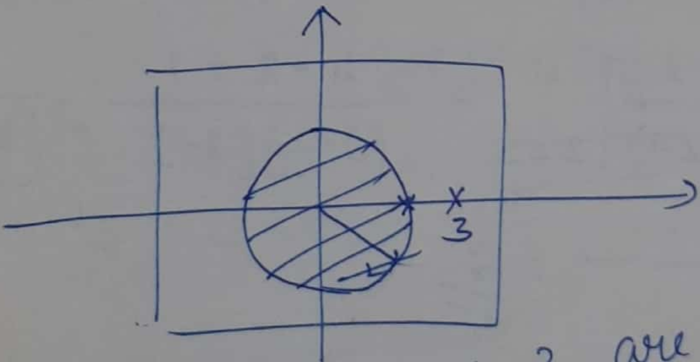
The poles at $z = 2$ & 3 are surrounded by ROC. hence we will have a +ve time seq^n (right sided seq.)

$$\therefore X(z) = \frac{z}{z-3} - \frac{z}{z-2}$$

IZT

$$x(n) = (3)^n u(n) - (2)^n u(n)$$

(2) ROC: $|z| < 2$



The poles at $z = 2$ & 3 are outside ROC hence we have left sided seq.

$$\therefore X(z) = \frac{z}{z-3} \overset{-}{\cancel{+}} \frac{z}{z-2}$$

$$x(n) = -(3)^n u(-n-1) + (2)^n u(-n-1)$$

② $$X(z) = \frac{1-z^{-1}+z^{-2}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)\left(1-z^{-1}\right)}$$

with ROC $1<|z|<2$

$1/2<|z|<1$

$$X(z) = \frac{1-\frac{1}{z}+\frac{1}{z^2}}{\left(\frac{z-1/2}{z}\right)\left(\frac{z-2}{z}\right)\left(\frac{z-1}{z}\right)} = \frac{\frac{z^2-z+1}{z^2}}{\frac{(z-1/2)(z-2)(z-1)}{z^3}}$$

$$= \frac{z(z^2-z+1)}{(z-1/2)(z-2)(z-1)}$$

$$\frac{X(z)}{z} = \frac{z^2-z+1}{(z-1/2)(z-2)(z-1)} = \frac{A}{(z-1/2)} + \frac{B}{(z-2)} + \frac{C}{(z-1}$$

$$A = \left.\frac{z^2-z+1}{(z-2)(z-1)}\right|_{z=1/2} = \frac{1/4-1/2+1}{(1/2-2)\left(\frac{1}{2}-1\right)} = \frac{3/4}{3/4} = 1$$

$$B = \left.\frac{z^2-z+1}{(z-1/2)(z-1)}\right|_{z=2} = \frac{4-2+1}{\left(2-\frac{1}{2}\right)(2-1)} = \frac{3}{\left(\frac{4-1}{2}\right)(1)}$$

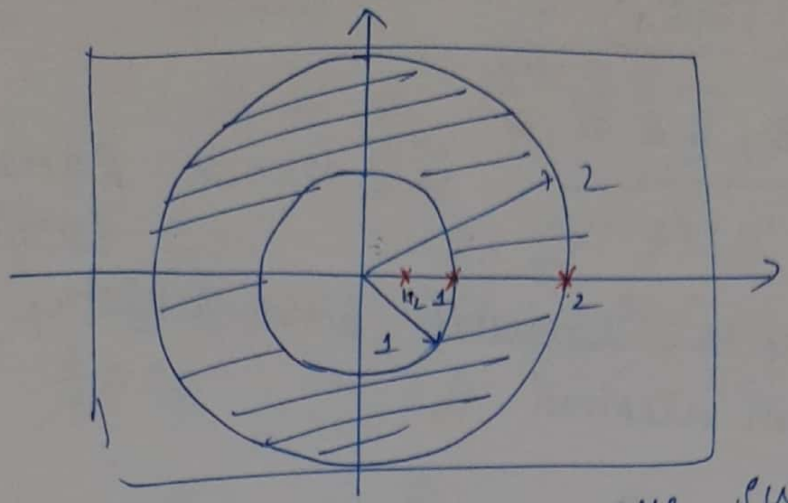
$$= \frac{3}{\frac{3}{2}} = \underline{\underline{2}}$$

$$C = \left.\frac{z^2-z+1}{(z-1/2)(z-2)}\right|_{z=1} = \frac{1-1+1}{(1-1/2)(1-2)} = \underline{\underline{-2}}$$

$$\therefore \frac{X(z)}{z} = \frac{1}{(z-1/2)} + \frac{2}{(z-2)} - \frac{2}{(z-1)}$$

$$X(z) = \frac{z}{(z-1/2)} + \frac{2z}{(z-2)} - \frac{2z}{(z-1)}$$

(i) ROC: $1 < |z| < 2$

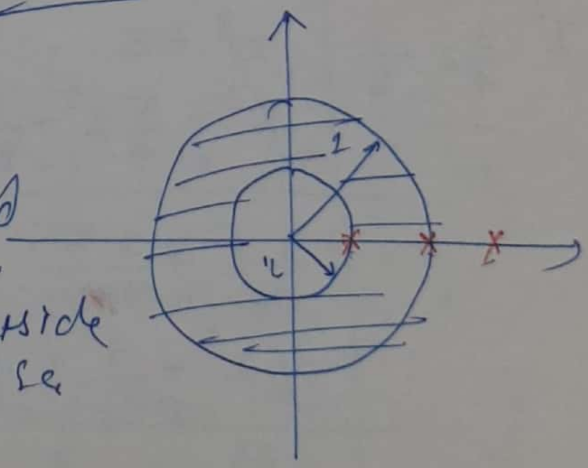


- poles at $z = 1/2$ & 1 are surrounded by ROC hence right sided seq.
- pole at $z = 2$ is outside ROC hence left sided seq.

$$\therefore \; X(z) = \frac{z}{z - 1/2} + \frac{2z}{z-2} - \frac{2z}{z-1}$$

$$x(n) = (1/2)^n u(n) - 2(2)^n u(-n-1) - 2(1)^n u(n)$$

(ii) $1/2 < |z| < 1$



- pole at 1/2 is surrounded by ROC hence right sided seq.

poles at $z = 1$ & 2 are outside ROC hence left sided seq.

$$x(n) : \left(\frac{1}{2}\right)^n u(n) - 2(2)^n u(-n-1) \cancel{+2u(n)}$$

$+ 2(1)^n u(-n-1)$

$+ 2u(-n-1)$

③ $X(z) = \frac{z^4 + z^2}{(z-\frac{1}{2})(z-1/4)} = z(z^3+z)$ ROC : $1/2 < |z| < \infty$

$$\frac{X(z)}{z} = \frac{z^3 + z}{(z-1/2)(z-1/4)}$$ is not a Rational fun

So to make it rational first perform long division method

$$\frac{z^3 + z}{z^2 + \frac{1}{8} - \frac{3}{4}z}$$

~~$z^2 - \frac{3}{4}z + \frac{1}{8}$) $z^3 + z^2$ (z~~

~~$\overset{(-)}{z^3} \overset{(+)}{-} \frac{3}{4}z^2 \overset{(-)}{+} \frac{1}{8}z$~~

$z^2 - \frac{3}{4}z + \frac{1}{8}$) $z^3 + z$ ($z + 3/4$

$\overset{(-)}{\cancel{z^3}} \overset{(+)}{-} \frac{3}{4}z^2 \overset{(-)}{+} \frac{1}{8}z$

$\frac{3}{4}z^2 + \frac{7}{8}z$

$\overset{(-)}{3/4\,z^2} \overset{(+)}{-} \frac{9}{16}z \overset{(-)}{+} \frac{3}{32}$

$\frac{23}{16}z - \frac{3}{32}$

$\frac{7}{8} + \frac{9}{16}$

$= \frac{14+9}{16}$

$$\frac{X(z)}{z} = \underbrace{z + \frac{3}{4}} + \frac{\frac{23}{16}z - \frac{3}{32}}{z^2 - \frac{3}{4}z + \frac{1}{8}}$$

$$Q + \frac{R}{D}.$$

$$\frac{\frac{23}{16}z - \frac{3}{32}}{z^2 - \frac{3}{4}z + \frac{1}{8}} = \frac{\frac{23}{16}z - \frac{3}{32}}{(z - 1/2)(z - 1/4)}$$

$$= \frac{A}{(z - 1/2)} + \frac{B}{(z - 1/4)}$$

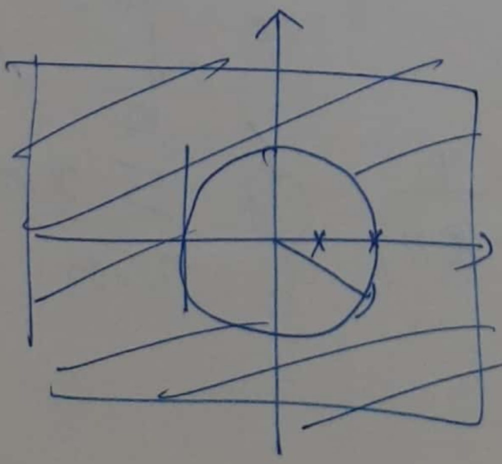
P7 4x
SP 4.18 - P3
43c

$$A = \left.\frac{\frac{23}{16}z - \frac{3}{32}}{(z - 1/4)}\right|_{z = 1/2} = \frac{5}{2}$$

$$B = \left.\frac{\frac{23}{16}z - \frac{3}{32}}{(z - 1/2)}\right|_{z = 1/4} = -\frac{17}{16}$$

$$X(z) = z^2 + \frac{3}{4}z + \frac{5}{2}\frac{z}{z - 1/2} - \frac{17}{16}\frac{z}{z - 1/4}$$

ROC. $1/2 < |z| < \infty$



poles at $z = 1/2$ & $1/4$
all surrounds by Ro

$$x(n) = \delta(n+2) + \frac{3}{4}\delta(n+1)$$
$$+ \frac{5}{2}(1/2)^n u(n)$$
$$- \frac{17}{16}\left(\frac{1}{4}\right)^n u(n)$$

~~Problems on Block and~~

extra problems on Z-Transform

(1) $x(n) = \left(\frac{1}{2}\right)^n \{u(n) - u(n-10))$

$$X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n \{u(n) - u(n-10)\} z^{-n}$$

$$= \sum_{n=0}^{9} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{9} \left(\frac{1}{2z}\right)^n$$

$$\sum_{n=0}^{N} (a)^n = \frac{1-a^{N+1}}{1-a}$$

$$= \frac{1 - \left(\frac{1}{2z}\right)^{10}}{1 - 1/2z}$$

$|z| > 0$

$|a| < 1$

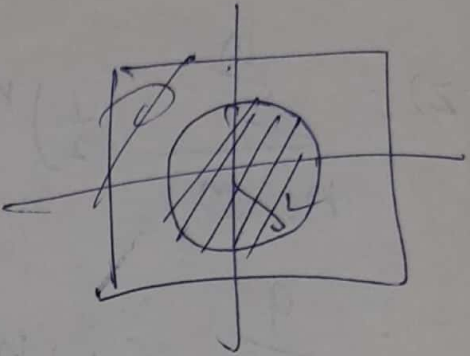
$\left|\frac{1}{2z}\right| < 1$

(2) $x(n) = 3(2)^n u(-n)$

$$X(z) = 3\sum_{n=-\infty}^{\infty} (2)^n u(-n) z^{-n}$$

$$= 3 \sum_{n=-\infty}^{0} (2)^{n} z^{-n}$$

replace n by -n



$$= 3 \sum_{n=0}^{\infty} 2^{-n} z^{n}$$

$$= 3 \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^{n}$$

$$= 3 \frac{1}{1 - \frac{z}{2}}$$

R.O.C.

$$\left|\frac{z}{2}\right| < 1$$

$$= \left(\frac{6}{2-z}\right)$$

$$|z| < 2$$

② $x(n) = (-1)^{n+1} u(n-1)$

$$X(z) = \sum_{n=-\infty}^{\infty} (-1)^{1} (-1)^{n} u(n-1) z^{-n}$$

$$= \sum_{n=1}^{\infty} (-1)^{1} (-1)^{n} z^{-n}$$

$$= -\sum_{n=1}^{\infty}\left(-\frac{1}{z}\right)^n$$

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

$|a| < 1$

$$= -\left[\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n - 1\right]$$

$$= -\left[\frac{1}{1+\frac{1}{z}} - 1\right]$$

$$= -\left[\frac{z}{z+1} - 1\right] = 1 - \frac{z}{z+1}$$

$\left|\frac{1}{z}\right| < 1$

$|1| < |z|$

$|z| > 1$

⑤ $x(n) = \alpha^{|n|}$

$x(n) = \alpha^n u(n) + \alpha^{-n} u(-n-1)$

$=$ [illegible] $-$

$$= \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{n=-\infty}^{-1} \alpha^{-n} z^{-n}$$

$$= \sum_{n=0}^{\infty}\left(\frac{\alpha}{z}\right)^n + \sum_{n=1}^{\infty}(\alpha z)^n$$

$$= \frac{z}{z-\alpha} + \frac{1}{1-\alpha z} - 1$$

$$= \frac{z}{z-\alpha} + \frac{1-1+\alpha z}{1-\alpha z}$$

$$= \frac{z}{z-\alpha} + \frac{\alpha z}{1-\alpha z}$$

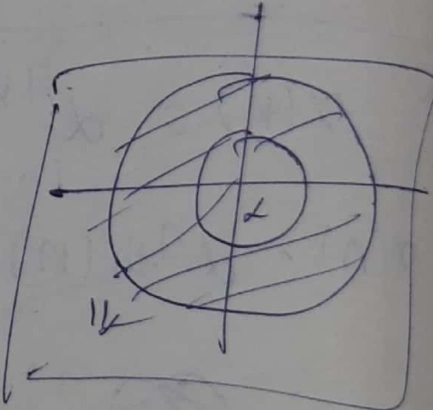
$$\left|\frac{\alpha}{z}\right| < 1 \cap |\alpha z| < 1$$

$$|\alpha| < |z| \cap (z) < \frac{1}{\alpha}$$

or

$$|z| > |\alpha| \cap (z| < \frac{1}{\alpha}$$

$$|\alpha| < |z| < \left|\frac{1}{\alpha}\right|$$



$$(n-1)^2 \left(\frac{1}{2}\right)^{n-1} u(n-1)$$

$$x(n) = x_1(n-1)$$

$$x_1(n) = n^2 \left(\frac{1}{2}\right)^n u(n)$$

$$= n^2 \cdot x_2(n)$$

$$= \cancel{n^2}\, x$$

④ $x(n) = 2(n-1)^2 \cdot \left(\frac{1}{2}\right)^n u(n-1)$

$= 2(n-1)^2 \left(\frac{1}{2}\right)^{n-1+1} u(n-1)$

$= \cancel{2}(n-1)^2 \left(\frac{1}{2}\right)^{n-1} \cdot \frac{1}{\cancel{2}} u(n-1)$

$? = (n-1)^2 \left(\frac{1}{2}\right)^{n-1} u(n-1)$

$= (n^2 - 2n + 1)$

$= n^2 \left(\frac{1}{2}\right)^{n-1} u(n-1) - 2n\left(\frac{1}{2}\right)^{n-1} u(n-1)$

$\frac{8z}{(2z-1)^2}$

$\frac{-z8}{(2z-1)^2}$

$\frac{2}{2z-1}$

$X(z) = \frac{8z^2+4z}{(2z-1)^3} - \frac{8z}{(2z-1)^2} + \frac{2}{2z-1}$

~~$2\left(\frac{1}{2}\right)^{i}$~~

an

SP4.5

ⓒ $x(n) = 3^{n+1} u(n) - 2(1/2)^n u(-n-1)$

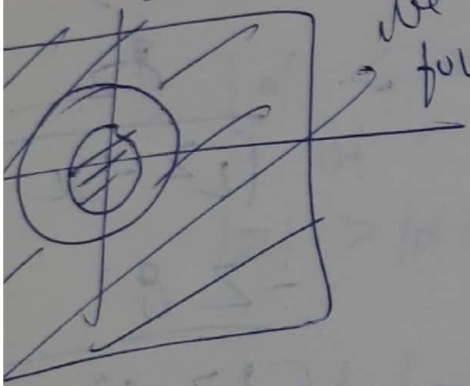
$= 3[3^n u(n)] - 2(\frac{1}{2})^n u(-n-1)$

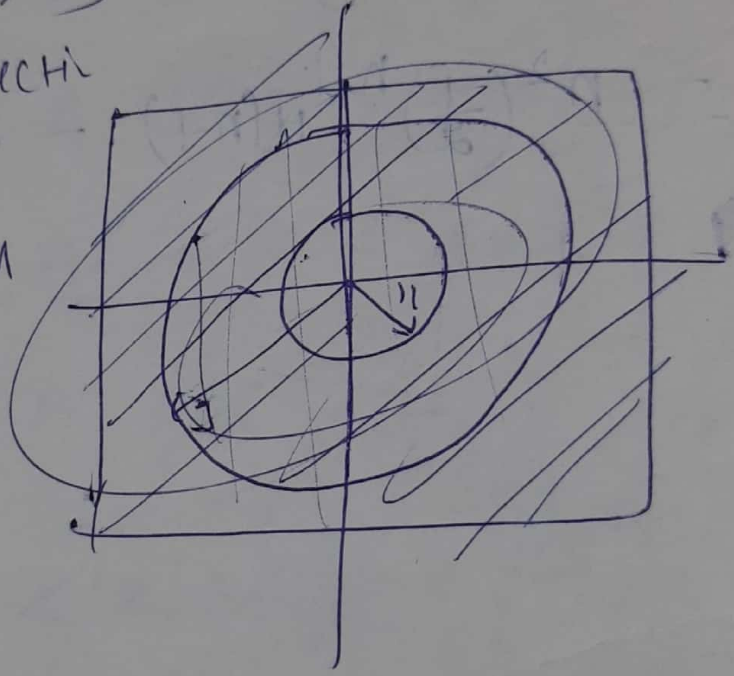
$= 3\left[\frac{z}{z-3}\right] + 2\left[\frac{z}{z-1/2}\right]$

$(|z|>3) \cap (|z| < \frac{1}{2}) =$ Nulln

No intersecti

∴ X(z) cannot be found





$$X(z) = \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

(1) determine the sequence $x(n)$ from foll fun

$$X(z) = \frac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}} \qquad x(n) \text{ is causal}$$

$$= \frac{z^2(1+z^{-1})}{z^2 - z + 0.5} = \frac{z^2+z}{z^2-z+0.5}$$

$$= \frac{z(z+1)}{z^2-z+0.5}$$

$$\frac{X(z)}{z} = \frac{(z+1)}{z^2-z+0.5} = \frac{z+1}{(z-\frac{1}{2}-j\frac{1}{2})(z-\frac{1}{2}+j\frac{1}{2}}$$

$$\frac{X(z)}{z} = \frac{A}{(z-\frac{1}{2}-j\frac{1}{2})} + \frac{B}{(z-\frac{1}{2}+j\,1/2)}$$

$$A = \left.\frac{z+1}{z-\frac{1}{2}+j\,1/2}\right|_{z=1/2+j\,1/2}$$

$$= \underline{\underline{\frac{1}{2} - j\frac{3}{2}}}$$

$$B = \frac{z+1}{\left(z - \frac{1}{2} - j\frac{1}{2}\right)}\Bigg|_{z = \frac{1}{2} - j\frac{1}{2}}$$

$$= \frac{1}{2} + j\frac{3}{2}$$

$$\boxed{A = B^*}$$

$$X(z) = \frac{\frac{1}{2} - j\frac{3}{2} \quad z}{1 - \left(\frac{1}{2} + j\frac{1}{2}\right)z^{-1}} + \frac{\frac{1}{2} + j\frac{3}{2} \quad z}{1 - \left(\frac{1}{2} - j\frac{1}{2}\right)z^{-1}}$$

$$x(n) = \left(\frac{1}{2} - j\frac{3}{2}\right)\left(\frac{1}{2} + j\frac{1}{2}\right)^n u(n)$$

$$+ \left(\frac{1}{2} + j\frac{3}{2}\right)\left(\frac{1}{2} - j\frac{1}{2}\right)^n u(n)$$

Example 4.16 Repeat Example 4.15 for the signal $x(n) = \left(\frac{1}{2}\right)^n u(n-2)$

Solution. Given : $x(n) = \left(\frac{1}{2}\right)^n u(n-2)$

We have,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u(n-2)z^{-n}$$

$$= \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$$

$$= \sum_{n=2}^{\infty} \left(\frac{1}{2}z^{-1}\right)^n \qquad \text{(E4.16.1)}$$

$$X(z) = \frac{\left(\frac{1}{2}z^{-1}\right)^2}{1 - \frac{1}{2}z^{-1}} \qquad \left[\because \sum_{n=k}^{\infty} \alpha^n = \frac{\alpha^k}{1-\alpha} \;;\; \text{if } |\alpha| < 1\right]$$

$$= \frac{1}{4z\left(z - \frac{1}{2}\right)}$$

Eqn. (E4.16.1) will converge if $\left|\frac{1}{2}z^{-1}\right| < 1$

i.e., $|z| > \frac{1}{2}$

The ROC and pole-zero locations are shown in Fig. E4.16.

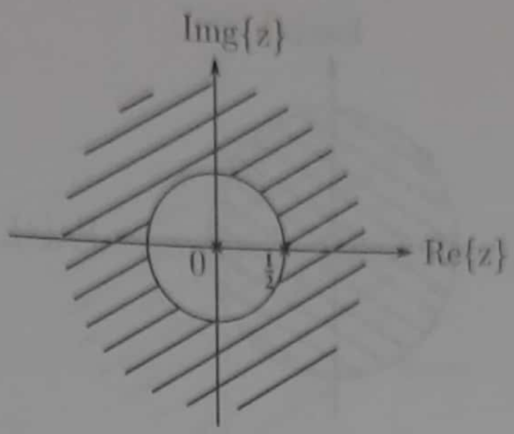


Fig. E4.16

Example 4.17 Repeat Example 4.15 for the signal $x(n) = 2^n u(-n-1)$.

Solution. Given: $x(n) = 2^n u(-n-1)$

We have,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} 2^n u(-n-1) z^{-n}$$

$$= \sum_{n=-\infty}^{-1} 2^n z^{-n}$$

$$= \sum_{n=-\infty}^{-1} \left(2z^{-1}\right)^n$$

$$= \sum_{n=1}^{\infty} \left(2^{-1}z\right)^n \qquad \text{(E4.17.1)}$$

$$= \frac{2^{-1}z}{1-2^{-1}z} \qquad \left[\because \sum_{n=1}^{\infty} \alpha^n = \frac{\alpha}{1-\alpha} \;;\; \text{if } |\alpha| < 1\right]$$

$$= \frac{-z}{z-2}$$

Eqn. (E4.17.1) will converge if $\left|2^{-1}z\right| < 1$

i.e., ROC : $|z| < 2$

The ROC and pole-zero locations are shown in Fig. E4.17.

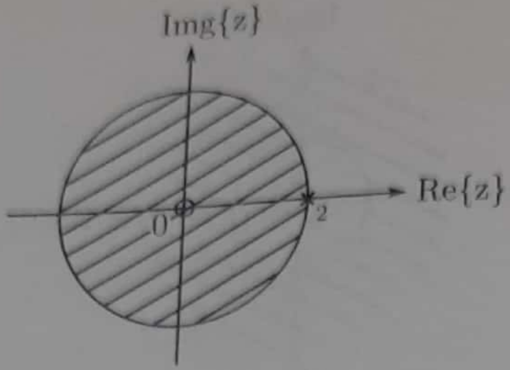


Fig. E4.17

Example 4.18 Repeat Example 4.15 for $x(n) = \left(\frac{1}{2}\right)^n \{u(n) - u(n-10)\}$

Solution. Given : $x(n) = \left(\frac{1}{2}\right)^n \{u(n) - u(n-10)\}$

We have,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^n$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n \{u(n) - u(n-10)\} z^{-n}$$

$$= \sum_{n=0}^{9} \left(\frac{1}{2}\right)^n z^{-n}$$

$$= \sum_{n=0}^{9} \left(\frac{1}{2} z^{-1}\right)^n \qquad \left[\begin{aligned} \sum_{n=0}^{N-1} \alpha^n &= \frac{1-\alpha^N}{1-\alpha} \quad ; \text{if } \alpha \neq 1 \\ &= N \quad ; \text{if } \alpha = 1 \end{aligned} \right.$$

$$= \frac{1 - \left(\frac{1}{2} z^{-1}\right)^{10}}{1 - \frac{1}{2} z^{-1}}$$

$$= \frac{1}{z^9} \frac{z^{10} - \left(\frac{1}{2}\right)^{10}}{z - \frac{1}{2}}$$

ROC: All z except z = 0.

The ROC and pole-zero locations are shown in Fig. E4.18.

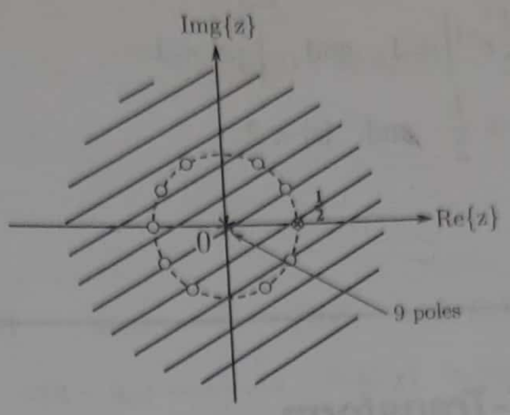


Fig. E4.18

Example 4.19 Find the Z-transform of $x(n) = \left(\frac{1}{2}\right)^{|n|}$ and find its ROC.

Solution. Given:

$$x(n) = \left(\frac{1}{2}\right)^{|n|}$$

$$= \left(\frac{1}{2}\right)^{n} u(n) + \left(\frac{1}{2}\right)^{-n} u(-n-1)$$

We have,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\left(\frac{1}{2}\right)^{n} u(n) + \left(\frac{1}{2}\right)^{-n} u(-n-1)\right] z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} z^{-n} + \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^{-n} z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^{n} + \sum_{n=1}^{\infty} \left(\frac{1}{2} z\right)^{n}$$

$$= \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{\frac{1}{2} z}{1 - \frac{1}{2} z}$$

$$\therefore \; X(z) = \frac{-\frac{3}{2} z}{(z-2)\left(z - \frac{1}{2}\right)}$$

In eqn. (E4.19.1), both the terms must converge

$$\text{i.e.,} \quad \left|\frac{1}{2}z^{-1}\right| < 1 \quad \text{and} \quad \left|\frac{1}{2}z\right| < 1$$

$$|z| > \frac{1}{2} \quad \text{and} \quad |z| < 2$$

$$\therefore \text{ROC}: \quad \frac{1}{2} < |z| < 2.$$

4.5.1.2 Partial Fraction Expansion of $X(z)$ with multiple poles

If X(z) (number of zeroes are lesser than poles) has multiple poles, the partial fraction is slightly different. For eg., consider that X(z) has 'N' poles. If the pole $z = \beta$ repeats 'L' times and the remaining 'N − L' poles are simple at $z = \psi_\ell \;;\; 1 \le \ell \le N - L$, then the partial fraction expansion is given by,

$$X(z) = \sum_{\ell=1}^{N-L} \frac{K_\ell}{(1 - \psi_\ell z^{-1})} + \sum_{i=1}^{L} \frac{\gamma_i}{(1 - \beta z^{-1})^{i}} \tag{4.18}$$

where the constants are computed using the formula,

$$\gamma_i = \frac{1}{(L - i)!(-\beta)^{L-i}} \frac{d^{L-i}}{d(z^{-1})^{L-i}} \left[\left(1 - \beta z^{-1}\right)^{L} X(z) \right] \Bigg|_{z=\beta} \quad ; 1 \le i \le L \tag{4.19}$$

and

$$K_\ell = \left(1 - \psi_\ell z^{-1}\right) X(z) \Big|_{z=\psi_\ell} \quad ;\; 1 \le \ell \le N - L \tag{4.20}$$

This would be better understood by looking at the following examples.

Example 4.49 Find the inverse Z-transform of,

$$X(z) = \frac{4 - 3z^{-1} + 3z^{-2}}{(1 + 2z^{-1})(1 - 3z^{-1})^2} \quad ; |z| > 3$$

Solution. Here there are 2 poles (i.e., $L = 2$) at $z = 3$ and $M < N$. The partial fraction expansion of $X(z)$ is,

$$X(z) = \frac{K_1}{(1+2z^{-1})} + \frac{\gamma_1}{(1-3z^{-1})} + \frac{\gamma_2}{(1-3z^{-1})^2}$$

Using eqn. (4.20),

$$K_1 = \left(1+2z^{-1}\right)X(z)\Big|_{z=-2} = \left.\frac{4-3z^{-1}+3z^{-2}}{(1-3z^{-1})^2}\right|_{z=-2} = 1$$

Using eqn. (4.19),

$$\gamma_1 = \frac{1}{(2-1)!(-3)^{2-1}}\frac{d^{2-1}}{d(z^{-1})^{2-1}}\left[\left(1-3z^{-1}\right)^2 X(z)\right]\Bigg|_{z=3}$$

$$= \frac{1}{(-3)}\frac{d}{d(z^{-1})}\left[\frac{4-3z^{-1}+3z^{-2}}{(1+2z^{-1})}\right]\Bigg|_{z=3}$$

$$= 1$$

$$\gamma_2 = \frac{1}{(2-2)!(-3)^{2-2}}\frac{d^{2-2}}{d(z^{-1})^{2-2}}\left[\left(1-3z^{-1}\right)^2 X(z)\right]\Bigg|_{z=3}$$

$$= \left[\frac{4-3z^{-1}+3z^{-2}}{(1+2z^{-1})}\right]\Bigg|_{z=3}$$

$$= 2$$

$$\therefore \quad X(z) = \frac{1}{(1+2z^{-1})} + \frac{1}{(1-3z^{-1})} + \frac{2}{(1-3z^{-1})^2}$$

Multiplying and dividing the 3rd term by $3z^{-1}$ we get,

$$X(z) = \frac{1}{(1+2z^{-1})} + \frac{1}{(1-3z^{-1})} + \frac{2\cdot 3z^{-1}z^{+1}}{3(1-3z^{-1})^2}$$

Since ROC: $|z| > 3$, $x(n)$ must be right sided sequence.

Referring to Table 4.1 (pairs 7 and 10) and using time-shift property to 3rd term,

$$x(n) = (-2)^n u(n) + (3)^n u(n) + \frac{2}{3}(n+1)(3)^{n+1}u(n+1)$$

4.6 Transform Analysis of LTI Systems

The Z-transform plays an important role in the analysis and representation of discrete-time LTI systems.

4.6.1 Transfer function and impulse response

Consider a discrete-time LTI system having impulse response h(n) as shown in Fig. 4.2.

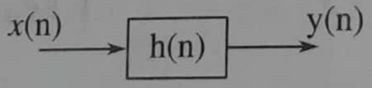


Fig. 4.2 A discrete-time LTI system

Let us say it gives output y(n) for the input x(n). Then we have,

$$y(n) = h(n) * x(n) \tag{4.23}$$

Taking Z-transform on both the sides we get,

$$Y(z) = H(z)X(z) \tag{4.24}$$

where $X(z) = Z -$ transform of the input $x(n)$.

$Y(z) = Z -$ tansform of the output $y(n)$.

$H(z) = Z -$ transform of the impulse response $h(n)$.

From eqn. (4.24) we get,

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.25}$$

In eqn. (4.25), H(z) is referred to as *system function* or *transfer function* of the system.

EXAMPLES

Example 4.73 A causal system has input $x(n)$ and output $y(n)$. Find the impulse response of the system if,

$$x(n) = \delta(n) + \frac{1}{4}\delta(n-1) - \frac{1}{8}\delta(n-2)$$

$$y(n) = \delta(n) - \frac{3}{4}\delta(n-1)$$

Solution. Given:

$$x(n) = \delta(n) + \frac{1}{4}\delta(n-1) - \frac{1}{8}\delta(n-2) \tag{E4.73.1}$$

and $$y(n) = \delta(n) - \frac{3}{4}\delta(n-1) \tag{E4.73.2}$$

Taking Z-transform of eqn. (E4.73.1) and (E4.73.2) we get,

$$X(z) = 1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}$$

$$Y(z) = 1 - \frac{3}{4}z^{-1}$$

∴ Transfer function or system function is given by,

$$H(z) = \frac{Y(z)}{X(z)}$$

$$= \frac{1 - \frac{3}{4}z^{-1}}{1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}$$

$$= \frac{1 - \frac{3}{4}z^{-1}}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)}$$

By partial fraction expansion we get,

$$H(z) = \frac{-\frac{2}{3}}{\left(1-\frac{1}{4}z^{-1}\right)} + \frac{\frac{5}{3}}{\left(1+\frac{1}{2}z^{-1}\right)} \tag{E4.73.3}$$

It is given that the system is causal. [i.e., h(n) = 0 for n < 0]
Taking the inverse Z-transform of eqn. (E4.73.3) we get,

$$\therefore \text{ Impulse Response } h(n) = \frac{-2}{3}\left(\frac{1}{4}\right)^n u(n) + \frac{5}{3}\left(\frac{-1}{2}\right)^n u(n)$$

$$= \frac{1}{3}\left[-2\left(\frac{1}{4}\right)^n + 5\left(\frac{-1}{2}\right)^n\right]u(n)$$

Example 4.74 We want to design a causal discrete-time LTI system with the property that if the input is $x(n) = \left(\frac{1}{2}\right)^n u(n) - \frac{1}{4}\left(\frac{1}{2}\right)^{n-1} u(n-1)$, then the output is $y(n) = \left(\frac{1}{3}\right)^n u(n)$.

Determine the impulse response h(n) and the system function H(z) of the system that satisfies this condition.

Solution. Given:

$$x(n) = \left(\frac{1}{2}\right)^n u(n) - \frac{1}{4}\left(\frac{1}{2}\right)^{n-1} u(n-1) \tag{E4.74.1}$$

$$y(n) = \left(\frac{1}{3}\right)^n u(n) \tag{E4.74.2}$$

Taking Z-transform of eqn. (E4.74.1) and eqn. (E4.74.2) we get,

$$X(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)} - \frac{\frac{1}{4}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)} = \frac{\left(1-\frac{1}{4}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)}$$

$$Y(z) = \frac{1}{\left(1-\frac{1}{3}z^{-1}\right)}$$

$$\therefore \text{ System function} \qquad H(z) = \frac{Y(z)}{X(z)}$$

$$\therefore \quad H(z) = \frac{\left(1 - \frac{1}{2}z^{-1}\right)}{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)}$$

Taking partial fraction expansion we get,

$$H(z) = \frac{-2}{\left(1 - \frac{1}{3}z^{-1}\right)} + \frac{3}{\left(1 - \frac{1}{4}z^{-1}\right)}$$

Taking inverse Z-transform we get,

$$\text{Impluse Response } h(n) = -2\left(\frac{1}{3}\right)^n u(n) + 3\left(\frac{1}{4}\right)^n u(n)$$

$$\therefore \quad h(n) = \left[3\left(\frac{1}{4}\right)^n - 2\left(\frac{1}{3}\right)^n\right] u(n)$$

Example 4.75 A system has impluse response $h(n) = \left(\frac{1}{2}\right)^n u(n)$. Determine the input to the system if the output is given by,

$$y(n) = \left(\frac{1}{3}\right) u(n) + \frac{2}{3}\left(-\frac{1}{2}\right)^n u(n)$$

Solution. Given:

$$y(n) = \left(\frac{1}{3}\right) u(n) + \frac{2}{3}\left(-\frac{1}{2}\right)^n u(n) \qquad \text{(E4.75.1)}$$

$$h(n) = \left(\frac{1}{2}\right)^n u(n) \qquad \text{(E4.75.2)}$$

We have

$$H(z) = \frac{Y(z)}{X(z)}$$

$$\therefore \quad X(z) = \frac{Y(z)}{H(z)}$$

Taking the Z-transform of eqn. (E4.75.1) and (E4.75.2) we get,

$$Y(z) = \frac{\frac{1}{3}}{(1 - z^{-1})} + \frac{\frac{2}{3}}{\left(1 + \frac{1}{2}z^{-1}\right)} = \frac{\left(1 - \frac{1}{2}z^{-1}\right)}{(1 - z^{-1})\left(1 + \frac{1}{2}z^{-1}\right)}$$

$$\& \quad H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

$$\therefore \quad X(z) = \frac{Y(z)}{H(z)}$$

$$= \frac{\left(1 - \frac{1}{2}z^{-1}\right)^2}{\left(1 - \frac{1}{2}z^{-1} - \frac{1}{2}z^{-2}\right)}$$

$$= \frac{1 - z^{-1} + \frac{1}{4}z^{-2}}{1 - \frac{1}{2}z^{-1} - \frac{1}{2}z^{-2}}$$

Since M $\nless$ N, we cannot go for partial fraction expansion directly.

By division we get,

$$\therefore \quad X(z) = -\frac{1}{2} + \frac{-\frac{5}{4}z^{-1} + \frac{3}{2}}{\left(1 - \frac{1}{2}z^{-1} - \frac{1}{2}z^{-2}\right)}$$

$$= -\frac{1}{2} + \frac{\frac{1}{6}}{(1 - z^{-1})} + \frac{\frac{4}{3}}{\left(1 + \frac{1}{2}z^{-1}\right)}$$

Taking inverse Z-transform we get,

$$x(n) = -\frac{1}{2}\delta(n) + \frac{1}{6}u(n) + \frac{4}{3}\left(-\frac{1}{2}\right)^n u(n)$$

$$x(n) = -\frac{1}{2}\delta(n) + \frac{1}{3}\left[\frac{1}{2} + 4\left(-\frac{1}{2}\right)^n\right]u(n)$$

4.6.2 Relationship between transfer function and difference equation

Consider a discrete-time LTI system for which the input and output satisfy a linear constant-coefficient difference equation of the form,

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \qquad (4.26)$$

Taking Z-transform on both the sides we get,

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$Y(z) \sum_{k=0}^{N} a_k z^{-k} = X(z) \sum_{k=0}^{M} b_k z^{-k}$$

$$\therefore \quad \text{Transfer function} \quad H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \qquad (4.27)$$

From eqn. (4.27), we can say that the transfer function of a system described by a linear constant-coefficient difference equation is a ratio of polynomials in z^{-1} and it is referred to as '*rational transfer function*'.

The coefficient of z^{-k} in the numerator polynomial is the coefficient associated with $x(n-k)$ and the coefficient of z^{-k} in the denominator polynomial is the coefficient associated with $y(n-k)$ in the difference equation.

EXAMPLES

Example 4.76 Find the transfer function and the impulse response of the system described by the difference equation,

$$y(n) - \frac{1}{2}y(n-1) = 2x(n-1)$$

Solution. Given:

$$y(n) - \frac{1}{2}y(n-1) = 2x(n-1)$$

Taking Z-transform on both the sides we get,

$$Y(z) - \frac{1}{2}z^{-1}Y(z) = 2z^{-1}X(z)$$

$$Y(z)\left[1 - \frac{1}{2}z^{-1}\right] = 2z^{-1}X(z)$$

$$\therefore \quad \text{Transfer function} \quad H(z) = \frac{Y(z)}{X(z)} = \frac{2z^{-1}}{1 - \frac{1}{2}z^{-1}} \qquad (E4.76.1)$$

Taking inverse Z-transform of eqn. (E4.76.1) we get,

Impulse response $h(n) = 2\left(\frac{1}{2}\right)^{n-1} u(n-1)$.

Example 4.77 Repeat Example 4.76 for,

$$y(n) - \frac{1}{4}y(n-1) - \frac{3}{8}y(n-2) = -x(n) + 2x(n-1)$$

Solution. Given: $y(n) - \frac{1}{4}y(n-1) - \frac{3}{8}y(n-2) = -x(n) + 2x(n-1)$

Taking Z-transform on both the sides we get,

$$Y(z)\left[1 - \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}\right] = X(z)\left[-1 + 2z^{-1}\right]$$

∴ Transfer function $H(z) = \frac{Y(z)}{X(z)} = \frac{-1 + 2z^{-1}}{1 - \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}$

$$H(z) = \frac{-1 + 2z^{-1}}{\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{3}{4}z^{-1}\right)}$$

By partial fraction expansion we get,

$$H(z) = \frac{-2}{\left(1 + \frac{1}{2}z^{-1}\right)} + \frac{1}{\left(1 - \frac{3}{4}z^{-1}\right)}$$

Taking inverse Z-transform we get,

Impulse response $h(n) = \left[-2\left(-\frac{1}{2}\right)^n + \left(\frac{3}{4}\right)^n\right]u(n).$

Example 4.78 A causal LTI system is described by the difference equation,

$$y(n) = y(n-1) + y(n-2) + x(n-1)$$

Find the system function H(z). Plot the poles and zeroes and indicate the ROC. Al determine the impulse response of the system.

Solution. Given: $y(n) = y(n-1) + y(n-2) + x(n-1)$

i.e., $y(n) - y(n-1) - y(n-2) = x(n-1)$

Taking Z-transform on both the sides we get,

$$Y(z) - z^{-1}Y(z) - z^{-2}Y(z) = z^{-1}X(z)$$

$$\therefore \quad \text{System function} \quad H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{(1 - z^{-1} - z^{-2})}$$

$$= \frac{z}{(z + 0.62)(z - 1.62)} \qquad \text{(E4.78.1)}$$

The poles and zeroes are shown in Fig. E4.78 below. Since the system is causal, the ROC must be outside the outermost pole. (i.e., z = 1.62)

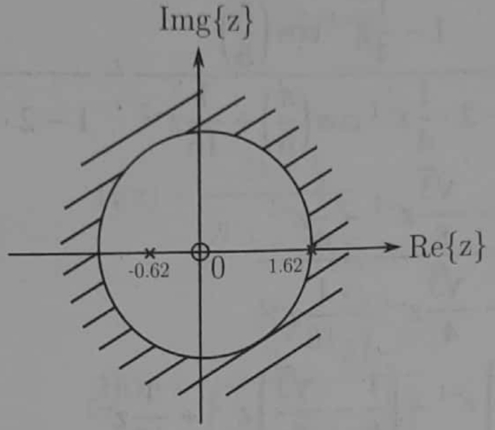


Fig. E4.78

From eqn. (E4.78.1)

$$H(z) = \frac{z^{-1}}{(1 + 0.62z^{-1})(1 - 1.62z^{-1})}$$

$$= \frac{K_1}{(1 + 0.62z^{-1})} + \frac{K_2}{(1 - 1.62z^{-1})}$$

$$K_1 = \left(1 + 0.62z^{-1}\right)H(z)\Big|_{z=-0.62} = \frac{z^{-1}}{(1 - 1.62z^{-1})}\Bigg|_{z=-0.62} = -0.45$$

$$K_2 = \left(1 - 1.62z^{-1}\right)H(z)\Big|_{z=1.62} = \frac{z^{-1}}{(1 + 0.62z^{-1})}\Bigg|_{z=1.62} = 0.45$$

$$\therefore \quad H(z) = \frac{-0.45}{(1 + 0.62z^{-1})} + \frac{0.45}{(1 - 1.62z^{-1})}$$

Taking inverse Z-transform we get,

$$h(n) = -0.45(-0.62)^n u(n) + 0.45(1.62)^n u(n)$$

Example 4.79 Find the transfer function and the difference equation representation for the system having impulse reponse,

$$h(n) = 2\left(\frac{2}{3}\right)^n u(n - 1) + \left(\frac{1}{4}\right)^n \left[\cos\left(\frac{\pi}{6}n\right) - 2\sin\left(\frac{\pi}{6}n\right)\right] u(n)$$

Solution. The given impulse response can be written as,

$$h(n) = \frac{4}{3}\left(\frac{2}{3}\right)^{n-1} u(n-1) + \left(\frac{1}{4}\right)^{n} \cos\left(\frac{\pi}{6}n\right)u(n) - 2\left(\frac{1}{4}\right)^{n} \sin\left(\frac{\pi}{6}n\right)u(n)$$

Taking Z-transform on both the sides,

$$H(z) = \frac{\frac{4}{3}z^{-1}}{1-\frac{2}{3}z^{-1}} + \frac{1-\frac{1}{4}z^{-1}\cos\left(\frac{\pi}{6}\right)}{1-2\cdot\frac{1}{4}z^{-1}\cos\left(\frac{\pi}{6}\right)+\frac{1}{16}z^{-2}} - \frac{2\cdot\frac{1}{4}z^{-1}\sin\left(\frac{\pi}{6}\right)}{1-2\cdot\frac{1}{4}z^{-1}\cos\left(\frac{\pi}{6}\right)+\frac{1}{16}z^{-2}}$$

$$= \frac{\frac{4}{3}z^{-1}}{1-\frac{2}{3}z^{-1}} + \frac{1-\frac{\sqrt{3}}{8}z^{-1}-\frac{1}{4}z^{-1}}{1-\frac{\sqrt{3}}{4}z^{-1}+\frac{1}{16}z^{-2}}$$

$$= \frac{1+\left(\frac{5}{12}-\frac{\sqrt{3}}{8}\right)z^{-1}+\left(\frac{1}{6}-\frac{\sqrt{3}}{4}\right)z^{-2}+\frac{1}{12}z^{-3}}{1-\left(\frac{2}{3}+\frac{\sqrt{3}}{4}\right)z^{-1}+\left(\frac{1}{16}+\frac{\sqrt{3}}{6}\right)z^{-2}-\frac{1}{24}z^{-3}} = \frac{Y(z)}{X(z)}$$

$$\therefore\quad y(n)-\left(\frac{2}{3}+\frac{\sqrt{3}}{4}\right)y(n-1)+\left(\frac{1}{16}+\frac{\sqrt{3}}{6}\right)y(n-2)-\frac{1}{24}y(n-3)$$

$$= x(n)+\left(\frac{5}{12}-\frac{\sqrt{3}}{8}\right)x(n-1)+\left(\frac{1}{6}-\frac{\sqrt{3}}{4}\right)x(n-2)+\frac{1}{12}x(n-3)$$

Stability & causality

A LTI system is causal if its impulse response satisfies the condition

$\boxed{h(n)=0} \quad n<0$

For a causal signal the ROC of the system fun (H(z)) is exterior of a circle of radius 'r'

If the system is causal, then the ROC for H(z) will be outside the outermost pole

If the system is stable, then its impulse response h(n) must be absolutely summable.

Alternatively, a causal system is stable if the poles of H(z) lies inside the unit circle in the z-plan

- For a system that is both stable & causal, the ROC must include the unit circle & it must be outside the outermost pole.

- For a system to be stable & causal, all the poles should lie inside the unit circle in the z-plane

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

$$|H(z)| \le \sum_{n=-\infty}^{\infty} |h(n) z^{-n}|$$

$$= \sum |h(n)| |z^{-n}|$$

when evaluated on unit cir $|z|=1$

$$\le \sum |h(n)|$$

∴ LTI system is BIBO stable if & only if the ROC of the system fun includes the unit circle

A ~~LTI~~ LTI system is characterized by system fun

$$H(z) = \frac{3-4z^{-1}}{1-\frac{7}{2}z^{-1}+\frac{3}{2}z^{-2}}$$

Specify the ROC of H(z) & determine h(n) for the foll

(a) The system is stable

(b) — " — causal

(c) — " — anticausal

Sol

$$H(z) = \frac{3-4z^{-1}}{1-\frac{7}{2}z^{-1}+\frac{3}{2}z^{-2}}$$

$$= \frac{3-4z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-3z^{-1}\right)}$$

$$= \frac{A}{(1-\frac{1}{2}z^{-1})} + \frac{B}{(1-3z^{-1})}$$

$A = 1$, $B = 2$

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{2}{1-3z^{-1}}$$

$$= \frac{z}{z-\frac{1}{2}} + 2\frac{z}{z-3}$$

poles are at $z = 1/2$ & $z = 3$

(a) for stability ROC must include the unit circle

$\therefore$ ROC $\frac{1}{2} < |z| < 3$

$\downarrow$

lies inside the un

$h(n) = \left(\frac{1}{2}\right)^n u(n) - 2(3)^n u(-n-1)$

(b) A system is causal if ROC is exterior to the G4

$|z| > 3$

$h(n) = \left(\frac{1}{2}\right)^n u(n) + 2(3)^n u(n)$

(c) $|z| < 1/2$

$h(n) = -\left(\frac{1}{2}\right)^n u(-n-1)$
$- 2(3)^n u(-n-1)$